# Logic, Methodology and Philisophy of Science at Warsaw University (2)

Edited by:
Anna Brożek, Jacek Juliusz Jadacki,
and Witold Strawiński

Wydawnictwo Naukowe *Semper*
Warszawa 2005

# Preface

This book is intended to be a continuation of the volume edited — three years ago — under the same title by Dr Mieszko Tałasiewicz. One of the main aims of publishing it is to present some results obtained in the title areas in the Institute of Philosophy of Warsaw University.

A novelty of this volume consists in the fact that we inserted in it some *inedita*. The first of these is Jan Łukasiewicz's ample text, "Theory of Deduction"; the second is Zdzisław Augustynek's short paper, "About Ontology".

The reason for publishing the first text is more than obvious. Łukasiewicz is the founder of the Warsaw School of Logic, a branch of the Lvov–Warsaw School, and all the contributors to this volume duly appreciate the great tradition of Kazimierz Twardowski's school.

"Theory of Deduction" is probably not the original title of Łukasiewicz's text: its typescript begins only with the "Introduction". What is more, this text was not finished by the author, or — precisely speaking — we have at our disposal only the part of it published in this volume. Łukasiewicz started to write this text in 1950. Some references to his project are contained in his correspondence with Rev. Józef M. Bocheński.[1] We enclose here some relevant excerpts from this correspondence (translated into English):

> 29.03.1946. When we acquire our own flat [in Dublin] — up to now we've been living in a boarding house — I shall start to write *Elements of Logistics* by order of the Parisian publishing firm *Les Presses Universitaires* (formerly: *Lacan*).[2]
>
> 4.09.1946. As concerns my handbook, I intend to put into it only ELEMENTS of mathematical logic, i.e., apart from the introduction (I): propositional calculus (II) — the proof of completeness and non-contradiction of this calculus (III) — the theory of quantifiers (IV) — the theory of predicates with the theory of identity (V) — Aristotle's syllogistics (VI) — and the application of the theory of natural numbers (VII). I do not consider the calculus of classes and relations because the theory of predicates replaces, in my opinion, both these calculi; besides, I do not like the theory of classes and the algebra of logic because I do not accept empty classes and I think that LEŚNIEWSKI's ontology is a much more perfect theory. I shall aspire to as great simplicity and clarity in presentation as possible — I am not satisfied with QUINE (*Math[ematical] Logic*) — TARSKI is more satisfying, but he gives too few theoretical comments. When I shall finish all these projects — I do not know. Up to now I have no conditions for free and intensive work. Writing will take at least one year.[3]
>
> 19.08.1950. Now I am starting to write a handbook of symbolic logic based on my lectures. I have not only lectures in the *Royal Irish Academy* twice a week (two full hours), but I have also delivered the cycle of eight lectures at *Queen's University* in Belfast, in February and March of this year.[4]

We publish Łukasiewicz's typescript with no editorial interventions (with the exception of a few evident corrections and interpolations, always marked by brackets).

The reason for publishing Augustynek's paper (found in his scholar heritage, donated to one of us) is also obvious — at least for the editors. Firstly, almost all

[1] Jan Łukasiewicz, "From the correpondence with J.M. Bocheński". [In:] *Logika i metafizyka* [Logic and Meaphysics]. *Miscellanea* edited by J.J. Jadacki. Warszawa 1998, WFiS UW, p. 513–529.

[2] *Ibidem*, p. 515.

[3] *Ibidem*, p. 519.

[4] *Ibidem*, p. 525.

of us take ourselves for Professor Augustynek's pupils. Secondly, many years ago, Professor Augustynek inaugurated the very fruitful scholarly cooperation between two departments of the Institute of Philosophy of Warsaw University: his Department of the Philosophy of Science and the Department of Logical Semiotics.

This book is one of the fruits of this cooperation.

* * *

We are very grateful to the Director of the Archives of Warsaw University for his kind permission to publish in this volume Jan Łukasiewicz's "Theory of Deduction".

*Anna Brożek*
*Jacek Juliusz Jadacki*
*Witold Strawiński*

JAN ŁUKASIEWICZ

# THEORY OF DEDUCTION

## [I] Introduction

### 1. Logic of terms and logic of propositions

Symbolic logic, like mathematics, is not a single system, but a set of several systems some of them differing from each other more then arithmetic from geometry. The main parts of symbolic logic are logic of terms and logic of propositions. I shall explain their difference on a simple example.

There exists in the traditional logic a law of the form

(1.1) Every *a* is *a*,

called the law of identity. There exists another law of identity, known already to the Stoics, of the form:

(1.2) If *p*, then *p*.

Let us describe the difference between these two laws of identity.

They differ first by the linguistic expressions "every–is" and "if–then". As we shall see later, both these expressions are proposition–forming functors of two arguments which are in our examples identical: in (1.1) *a* and *a*, in (1.2) *p* and *p*. I call both expressions proposition–forming functors, because from their combination with the arguments results in both cases a proposition. A second and more important difference between these laws lies in their arguments: both arguments, *a* and *p*, are variables; that means, they have no determinate meaning, but are connected with a range of values which can be substituted instead of them. The values of *a* are universal terms, as "man" or "philosopher". By substituting in (1.1) instead of *a* the value "man", we get the proposition:

(1.3) Every man is a man.

The values of *p* are propositions, as "today is Tuesday" or "3 is less than 5". By substituting in (1.2) instead of *p* the value "today is Tuesday", we get the proposition:

(1.4) If today is Tuesday, then today is Tuesday.

Terms and propositions are two different semantical categories. You can significantly replace a proposition by another proposition, or a term by another term, but you cannot significantly put a proposition instead of a term, or vice–versa. For instance, because *p* denotes a proposition, and the law of identity (1.1) is a proposition, you can significantly substitute instead of *p* the proposition "every *a* is *a*", getting thus a new proposition:

(1.5) If every *a* is *a*, then every *a* is *a*.

But you cannot substitute instead of the term–variable *a* the proposition "if *p*, then *p*", since you would get a nonsens:

(1.6) Every if *p*, then *p* is if *p*, then *p*.

Logical expressions containing term–variables belong to the logic of terms, expressions containing only propositional variables belong to the logic of propositions. The law of identity (1.1) belongs to the logic of terms, the law of identity (1.2) to the logic of propositions. The difference between these two laws, and consequently the difference between the logic of terms and the logic of propositions, is as great as the semantical difference between terms and propositions, and therefore a fundamental one. This difference is greater than between arithmetic and geometry, because in the latter case we have only different kinds of terms, arithmetical terms denoting numbers, geometrical – spatial entities.

**2. Some historical remarks**

The most fundamental logical system is not the logic of terms, but the logic of propositions. Historically, however, the logic of terms arose earlier than the logic of propositions. The earliest logical system is the syllogistic of Aristotle (384–322 B.C.), and this system belongs to the logic of terms.

All Aristotelian syllogisms are propositions beginning with "if", we call today such propositions "implications", and consist of two premisses and the conclusion. The premisses are connected by means of the conjunction "and". The most famous syllogism is the following called in the Middle Ages *Barbara*:

If every *a* is *b*,
(2.1) and every *b* is *c*,
then every *a* is *c*.

The letters *a*, *b* and *c* are term–variables, and the range of their values are universal terms, i.e. terms that may be predicated of more than one subject. Take for instance for *a* — "philosopher", for *b* — "man", and for *c* — "mortal": you get a syllogism in concrete terms:

If every philosopher is a man,
(2.2) and every man is mortal,
then every philosopher is mortal.

Aristotle has carefully studied all syllogistic forms and has constructed a system which was dominant through many centuries being till today the kernel of the so called formal logic. He has known only a few laws of propositional logic, not being aware that these laws belong to a logical system more fundamental than his own, and are indispensable for proving some syllogistic moods by means of syllogisms assumed by him axiomatically. He uses them in his proofs intuitively. The discovery of the propositional logic was reserved for the Stoics and their prominent logician Chrysippus (III century B.C.).

The logic of the Stoics is a logic of propositions. The most known Stoic syllogism, called later *modus ponens*, reads:

If $\alpha$, then $\beta$.
(2.3) But $\alpha$.
Therefore $\beta$.

The letters $\alpha$ and $\beta$ are propositional variables. Put for $\alpha$ – "today is Tuesday", and for $\beta$ — "tomorrow is Wednesday"; you get a syllogism in concrete terms:

If today is Tuesday, then tomorrow is Wednesday.
(2.4) But today is Tuesday.
Therefore tomorrow is Wednesday.

The Stoics have found a lot of such arguments all belonging to the logic of propositions. These arguments were known to logicians of the Middle Ages, and were even developed by them in their treatises *De consequentiis*, but were always treated as an appendix to the Aristotelian syllogistic. The modern system of the logic of propositions has been created only in 1879 by the great German logician Gottlob Frege in his paper

*Begriffsschrift*. Another outstanding logician of the XIX century, the American Charles Sanders Peirce, has greatly contributed to this logic by his discovery of logical matrices (1885). The authors of the *Principia mathematica*, Alfred North Whitehead and Bertrand Russell, have later put this system of logic on the top of all mathematics under the title: *Theory of Deduction* (1910).

**3. Theses and rules of inference**

The comparison of the Aristotelian syllogism (2.2) with the Stoic syllogism (2.3) is in another respect very instructive. The Aristotelian syllogism is an implication, therefore a proposition, and a true one. True propositions of a deductive system I shall call theses. The Stoic syllogism is not a proposition; it consists of three propositions, (I) if $\alpha$, then $\beta$, (II) $\alpha$, and (III) $\beta$, which are not unified so as to form one single proposition. The two premisses, (I) and (II), are stated without a conjunction, and the connexion of these loose premisses with the conclusion (III) by means of "therefore" does not give a new compound proposition. The Stoic syllogism (2.3) is not a proposition, it is an inference. Inferences are always recognizable by the word "therefore". Inferences, not being propositions, are neither true nor false, as truth and falsity belong only to propositions. They may be valid or not. The Stoic syllogism (2.3) is a valid inference, and so is (2.4). The syllogism (2.4) is an inference in concrete terms, the syllogism (2.3) I call a rule of inference, because it is stated in variables. The sense of this rule may be explained thus: When you put such values for $\alpha$ and $\beta$ that the premisses: if $\alpha$, then $\beta$, and $\alpha$ are true, then you must accept as true the conclusion $\beta$.

Every system of logic consists of theses and rules of inference. The proposition "every *a* is *a*" is a thesis of the Aristotelian syllogistic, the proposition "if *p*, then *p*" a thesis of the logic of propositions. The Stoic syllogism (2.3) is a rule of inference. Rules of inference are indispensable in any logical system. The chief purpose of logic is to derive new true propositions from already established truths. This can be only done by transforming given true propositions into new ones according to some very carefully formulated rules that lead from truth to truth. There is a tendency to reduce the number of the rules of inference to a minimum, by replacing them by theses. But some rules of inference must always remain, because otherwise we could not go forward. One of the most important rules of inference accepted in modern formal logic is just the Stoic syllogism *modus ponens*. It is called today the rule of detachment, because it enables us to detach and accept as true the consequent of an implication, provided the whole implication and its antecedent are true.

**4. Functors and arguments**

In order to explain what is meant by the words "functor" and "argument" let us consider some examples borrowed from elementary mathematics and logic.

A sum of two numbers, for instance:

(4.1) $3 + 5$,

is denoted by three symbols, 3, +, 5, and forms a whole which denotes also a number. The symbol which makes of the above expression a meaningful whole is the sign of addition, +. I call it the functor, and the other two symbols, 3 and 5, its arguments. Because both arguments as well as their sum denote numbers, I say that + is a number–forming functor of two numerical arguments. Let us now consider the expression:

(4.2) $3 < 5$.

This expression consists again of three symbols, 3, $<$, 5, but the whole formed by them does not denote a number; it is a proposition. The symbol which makes of the expression (4.2) a meaningful whole is the sign of the relation "less than", $<$. This sign is the functor, and 3 and 5 are its arguments. As the arguments denote numbers, and the whole is a proposition, I say that $<$ is a proposition–forming functor of two numerical arguments. As a third example I take the expression:

(4.3) If $3 < 5$, then $5 > 3$.

The whole formed by this expression is an implication, therefore a proposition. The sign upon which depends this whole are the words "if–then". This is the functor, and the two propositions, $3 < 5$ and $5 > 3$, are its arguments. The first proposition, $3 < 5$, is the antecedent, the second proposition, $5 > 3$, is the consequent. Because both arguments as well as their whole are propositions, I call the functor "if–then" a proposition–forming functor of two propositional arguments.

There exist functors of more than two arguments, and also functors of only one argument. An important proposition–forming functor of one propositional argument is the negation–functor. If we say for instance:

(4.4.) It is not true that $5 < 3$,

we get the negation of the proposition $5 < 3$, where the expression "it is not true that" is the functor, and the proposition $5 < 3$ the argument.

The distinction between functors and arguments is very useful when we want to introduce a reasonable symbolic notation into logic or mathematics.

### 5. Symbolic notation

Symbolic logic tends to attain the greatest possible exactness. In order to reach this aim it is more convenient to employ a special symbolism invented for this purpose, than to make use of ordinary language having its own grammatical laws. I shall now describe a symbolism I have invented and employed in my logical papers since more than twenty years which is, in my opinion, the simplest and the most reasonable one.

My symbolic notation which can be applied to logic as well as to mathematics is based on the following three principles:

(a) As simple symbols I am always using, besides numerical figures, small and capital letters of different kind and shape, Latin, Greek and German letters. Such symbols are available in every printers office.

(b) Simple symbols I am always arranging in straight rows in order to build up of them compound symbols.

(c) I am always writing the functors immediately before their arguments.

This third principle is very important, for it enables me to avoid brackets. Let us explain this point on an example borrowed from mathematics.

The associative law of addition runs in the ordinary notation thus:

(5.1) $(a+b)+c = a+(b+c)$

and cannot be stated without brackets. If you omit the brackets, you get the formula:

(5.2) $a+b+c = a+b+c$,

which does not represent the law of association, being only – if senseful at all – a special case of the law of identity:

(5.3) $a = a$.

In my notation I replace first the symbol + by the capital German $\mathfrak{P}$(sum), as I am using German capitals for denoting number–forming functors of numerical arguments, and secondly I replace the symbol = by the Latin capital $\mathcal{E}$ in its written form, as I am using such letters for denoting proposition–forming functors of numerical arguments. Instead of $(a+b)+c$ I write $\mathcal{P}\,\mathcal{P}\,abc$, instead of $a+(b+c)$ I write $\mathcal{P}\,a\,\mathcal{P}\,bc$, and the law of association gets thus the form:

(5.4) $\mathcal{E}\,\mathcal{P}\,\mathcal{P}abc\mathcal{P}a\mathcal{P}bc$.

This formula contains no brackets, is shorter than the usual formula (5.1), and can be read only in one way. It is the most concise expression of the associative law of addition.

In the same way I am writing logical formulae. In the *Principia mathematica* the implicational functor is denoted by a symbol similar to an inverted C which is written between its arguments. The implication "if $p$, then $q$" is thus denoted by $p \supset q$. The compound expression $p \supset q \supset r$ without brackets or dots is senseless or at least ambiguous, for it can mean either $(p \supset q) \supset r$ or $p \supset (q \supset r)$. In my notation no expression can be ambiguous. Using $C$ for denoting the implicative functor "if" I write $CCpqr$ instead of $(p \supset q) \supset r$, and $CpCqr$ instead of $p \supset (q \supset r)$. The expression $p \supset q \supset r$ cannot be written at all, it is senseless. In the same manner I write all the other propositional functions of two arguments, as conjunctions, alternations and equivalences. Each functor of two arguments is placed on the front of its arguments and is immediately succeeded by them. This must be kept in mind, when you want to understand a more complicated formula, e.g. the so called law of the hypothetical syllogism which has the form:

(5.5) $CCpqCCqrCpr$.

Knowing that C is a functor of two propositional arguments which immediately succeed C forming together with C a new compound propositional expression, you must first find out the simple implications, i.e. the implications having as arguments propositional variables. Of such a kind are the expressions $Cpq$, $Cqr$, and $Cpr$, contained in the above formula. Draw brackets around each of them: you will get the expression:

(5.6) $C(Cpq)C(Cqr)(Cpr)$.

Now you can easily see that the initial C of the formula has the propositional expression $(Cpq)$ as its first argument, and the whole rest, i.e. $C(Cqr)(Cpr)$ as its second argument. This rest is again an implication having $(Cqr)$ as its first, and $(Cpr)$ as its second argument. In a similar way you can analyze all the formulae containing C or other functors of two propositional arguments.

If there is a formula with a negation – I denote the negation–functor by $N$ – you must keep in mind that negation is a function of one argument only, which immediately succeeds $N$ and forms together with $N$ a new propositional expression. Having therefore a formula with $N$, for instance:

(5.7) $CpCNpq$,

you must, if you want to understand this formula, draw brackets around the simple negation $Np$, thus:

(5.8) $CpC(Np)q$,

and then you can see at once that the whole formula is an implication with $p$ as its first argument, and with $C(Np)q$ as its second argument. The last expression is also an implication having the negation $(Np)$ as its first and the variable $q$ as its second argument.

**6. What is formal logic**

The symbolism described roughly in the preceding section enables us to formalize logical and mathematical proofs. But before explaining what is meant by formalization we must know what is formal logic, since symbolic logic is formal logic.

"It is usual to say that logic is formal, in so far as it is concerned merely with the form of thought, that is with our manner of thinking irrespective of the particular objects about which we are thinking." This is a quotation from the well–known textbook of *Formal Logic* by J.N. Keynes. I read in this quotation the expression "form of thought" which I do not understand. Thought is a psychical phenomenon and psychical phenomena have no extension. I wonder what is meant by the form of an object which has no extension. The expression "form of thought" is inexact and it seems to me that this inexactitude arose from a wrong conception of logic. If you believe indeed that logic is the science of the laws of thought, you will be disposed to think that formal logic is an investigation of the form of thought.

It is not true however that logic is the science of the laws of thought. It is not the object of logic to investigate how we are thinking actually or how we ought to think. The first task belongs to psychology, the second to a practical art of a similar kind as mnemonics. Logic has no more to do with thinking than mathematics has. You must think, of course, when you have to perform an inference or a proof, as you must think too, when you have to solve a mathematical problem. But the laws of logic do not concern your thinking in a greater degree than do mathematical laws. The so called "psychologism" in logic is a mark of the decay of logic in modern philosophy.

The problem what is formal logic arose on the basis of the Aristotelian syllogistic. Let us consider the syllogism (2.2); this syllogism contains three concrete terms, "philosopher", "man", and "mortal". These terms are called the matter of the syllogism. But concrete terms, as "philosopher" or "man", do not belong to logic, since logic is not a science about philosophers or men. If we want to get a pure logical law, we must remove the matter from the syllogism. This was done by Aristotle, who has introduced variables instead of concrete terms. The syllogism (2.1) is a pure logical law, because it contains no matter, but can be applied to various kinds of matter. What remains after the matter of the syllogism has been removed, is called the form of the syllogism. Let us see of what elements consists this form.

To the form of the syllogism belong, besides the number and the disposition of variables, the so called logical constants. These constants are in our case logical functors. Two of them, the conjunctions "and" and "if–then", are proposition–forming functors of two propositional arguments, and belong to the logic of propositions. The third functor, "every–is", is a proposition–forming functor of two terminal arguments, and is characteristic for the Aristotelian syllogistic. There are still three other functors of this kind in the syllogistic, viz. "no–is", "some–is", and "some–is–not". All these four constant functors, being functors of two arguments, are relations in the field of universal terms. The Aristotelian syllogistic is the theory of these relations, and it is obvious that such a theory has nothing more in common with our thinking, than for instance the theory of the relations greater and less in the field of numbers. In a similar way the logic of propositions is the theory of logical constants in the domain of propositions. All logical systems are formal in this sense, that they employ variables, term– or proposition–variables, instead of their concrete values, and are investigating the properties of logical constants, especially functors, by means of which these variables are connected.

**7. Formalization**

It is obvious that the greatest possible exactness which is the aim of modern formal logic, can be reached only by means of a precise language built up of stable, visually perceptible signs. Such a language is indispensable for any science. Our own thoughts not formed in words are for ourselves almost inapprehensible, and the thoughts of other people, when not bearing an external shape, could be accessible but to a clairvoyant. Every scientific truth, in order to be perceived and verified, must be put into an external form intelligible to everybody. All these statements seem incontestably true. Modern formal logic gives therefore its utmost attention to the precision of language. The so called formalism is the consequence of this tendency. In order to understand what it is, let us analyze the following example.

I take as example the Stoic syllogism (2.3) which is a rule of inference, called formerly *modus ponens* and now the rule of detachment. According to this rule, if an implication of the form "if $\alpha$, then $\beta$" is asserted and the antecedent $\alpha$ of this implication is asserted too, we are allowed to assert its consequent $\beta$. In order to be able to apply this rule we must know that the proposition $\alpha$, asserted separately, expresses "the same" thought as the antecedent $\alpha$ of the implication, since only in this case we are allowed to perform the inference. We can state this only in the case, when these two $\alpha$'s have exactly the same external form, i.e. when they are equiform. For we cannot directly grasp the thoughts expressed by these $\alpha$'s and a necessary, although not sufficient condition for identifying two thoughts is the equiformity of their expressions. When, for instance, asserting the implication: "if all philosophers are men, then all philosophers are mortal" you would assert as the second premiss the sentence "every philosopher is a man", you could not get from these premisses the conclusion: "all philosophers are mortal", because you would have no guarantee that the sentence "every philosopher is a man" represents the same thought as the sentence "all philosophers are men". It would be necessary to confirm by means of a definition, that "every *a* is *b*" means the same as "all *a*'s are *b*'s", replace on the ground of this definition the sentence "every philosopher is a man" by the sentence "all philosophers are men", and only then it would be possible to get the conclusion. By this example you can easily comprehend the meaning of formalism. Formalism requires that the same thought should always be expressed by means of exactly the same series of words or symbols ordered in exactly the same manner. When a proof is formed according to this principle, we are able to control its validity on the basis of its external form only, without referring to the meaning of terms used in the proof. In order to get the conclusion $\beta$ from the premisses "if $\alpha$ then $\beta$" and $\alpha$, we need not know either what $\alpha$ or what $\beta$ really means; it suffices to notice that the two $\alpha$'s contained in the premisses are equiform.

A proof is full and formalized, if it has no gaps and if every its step is done according to a previously established rule of inference. To control such a proof it is sufficient to know the rules of inference applied to the proof.

**8. Examples of formalized proofs**

I shall give here two examples of formalized proofs borrowed from elementary mathematics. These examples will show the role played in proofs by the rule of inference on the one side, and by logical theses on the other side.

First example: We have to prove that no number is less than itself starting from two premisses: any number is equal to itself, and if two numbers are equal to each other, then it is not true that the one is less than the other.

We must first express the premisses and the conclusion in a precise symbolic language. Using the letter $\mathcal{E}$ to denote the relation of equality, the letter $\mathcal{L}$ to denote the relation "less than", the letter C to denote the functor "if–then", the letter $N$ to denote negation, and employing the letters $a$ and $b$ as numerical variables, we get as premisses the formulae:

(8.1) $\mathcal{E}aa$,

(8.2) $C\mathcal{E}abN\mathcal{L}ab$.

The first premiss reads in words: "$a$ equals $a$", the second: "if $a$ equals $b$, then it is not true that $a$ is less than $b$". We have to draw the conclusion:

(8.3) $N\mathcal{L}aa$,

that means: "it is not true that $a$ is less than $a$".

I shall apply to this proof two rules of inference: the rule of detachment and the rule of substitution. The first rule has been already explained. The second rule I shall use in its simplest form, viz.: it is allowed to get from an asserted thesis a new one, by substituting in every place where a variable occurs in it, another variable of the same kind. For instance, I shall substitute in (8.2) $a$ for $b$ in every place where $b$ occurs. I note this substitution in the following way:

(8.2) $b/a$ * (8.3)

I call this line the derivational line, for it precedes and justifies the subsequent new thesis (8.3). The symbol "/", used only in derivational lines, is the sign of substitution, * is a mark dividing the formula into two parts. The first part: (8.2) $b/a$, means that we have to put in (8.2) instead of the variable $b$ the variable $a$, the second part: (8.3) denotes the new thesis arising by this substitution. It is the thesis:

(8.3) $C\mathcal{E}aa\mathcal{L}aa$.

In words: "If $a$ equals $a$, then it is not true that $a$ is less than $a$". To this thesis we apply the rule of detachment. (8.3) is an implication, because it begins with a C, and has as antecedent the asserted proposition (8.1), $\mathcal{E}aa$. We are allowed therefore to detach its consequent $N\mathcal{L}a$, and assert it separately. I write again a derivational line which justifies this consequence:

(8.3) * C(8.1)–(8.4)

The two parts of this line denote the same thesis (8.3). The second part shows how this thesis is constructed, making obvious that the rule of detachment may be applied to it: (8.3) is an implication with (8.1) as antecedent, so that we may detach (8.4) as a new thesis. The symbol "–", used only in derivational lines, is the sign of detachment. We get therefore:

(8.3) $N\mathcal{L}aa$.

The whole proof consists of six lines which I put here together;

(8.1) $\mathcal{E}aa$

(8.2) $C\mathcal{E}abN\mathcal{L}ab$

(8.2) $b/a$ * (8.3)

(8.3) $C\mathcal{E}aaN\mathcal{L}a$

(8.3) * C(8.1)–(8.4)

(8.4) $N\mathcal{L}aa$

This proof has no gaps, it is therefore a full proof, and it is a formalized one, as you can check its validity knowing only the rules of inference. It suffices to state first: that (8.3) differs from (8.2) only in this respect, that in place of $b$ occurs everywhere $a$, and secondly: that the initial C of (8.3) is immediately followed by a set of three letters equiform to (8.1). You need not know the meaning of *Ɛ* and of *∠* either. The proof is valid for any *Ɛ* and *∠* satisfying the premisses. If, for instance, the values for *a* and *b* are expressions denoting lines on a plane, and *Ɛab* means "*a* is parallel to *b*", and *∠ab* means "*a* is perpendicular to *b*", the proof will be likewise valid. For it is true that any line is parallel to itself, and if two lines are parallel to each other, then it is not true that the one of them is perpendicular to the other. From these premisses results the conclusion that no line is perpendicular to itself.

Second example. We have again to prove that no number is less than itself starting from only one premiss: if *a* is less than *b*, it is not true that *b* is lees than *a*. In symbols:

(8.5) *C∠abN∠ba*.

This premiss however, together with the rules of substitution and detachment, is not sufficient to prove the required conclusion. We must still have an auxiliary premiss belonging to the logic of propositions. It is the thesis:

(8.6) *CCpNpNp*.

In words: "if (if *p*, then it is not true that *p*), then it is not true that *p*". The antecedent of (8.6) is *CpNp*, the consequent is *Np*. *Np* is the negation of *p*, and two such propositions, as *p* and *Np*, are called contradictory. The principle of excluded contradiction, stated by Aristotle, says that the conjunction of two contradictory propositions, "*p* and *Np*", is never true. The antecedent of our thesis, however, is not a conjunction, but an implication, and an implication of the form *CpNp* can be true provided that *p* is false and *Np* is true. A contradiction would arise in this case, when asserting *CpNp* we would also assert *p*; for from *CpNp* and *p* results by the rule of detachment *Np*, and two contradictory propositions, *p* and *Np*, would be together true. We must therefore accept that from *CpNp* results *Np*, and this is the sense of our thesis (8.6). It is a kind of *reductio ad absurdum*.

The formalized proof of *N∠aa* based on the premisses (8.5) and (8.6) runs thus:

(8.5) *C∠abN∠ba*

(8.6) *CCpNpNp*

  (8.6) *p* / *∠aa* * (8.7)

(8.7) *CC∠aaN∠aaN∠aa*

  (8.5) *b* / *a* * (8.8)

(8.8) *C∠aaN∠aa*

  (8.7) * C(8.8)–(8.9)

(8.9) *N∠aa*

I hope that this proof will be intelligible to all readers who have carefully studied the first example. As *p* is a propositional variable, we can substitute *∠aa* for *p* getting thus the thesis (8.7). The other two steps of the proof do not involve a difficulty. Thesis *CCpNpNp* which belongs to the logic of propositions is essential for this proof. It would be impossible to get from the mathematical premiss (8.5) the conclusion (8.9) by means of the Aristotelian syllogistic. Almost all mathematical proofs require some auxiliary theses belonging to propositional logic. It is therefore highly important that every student of mathematics and logic should know the logic of propositions.

# [II] A Survey of Theses*

Theory of deduction is the most elementary part of the logic of propositions. It contains only propositional variables and proposition–building functors of propositional expressions. We shall later extend this system by introducing constant arguments, variable functors, and quantifiers. As the theory of deduction is not sufficiently known, I shall give before a systematic exposition of this theory** a survey of its most important theses together with their mutual relations and some "meta–logical" explanations. I shall start with implicational theses, i.e. with theses containing besides propositional variables only the implicational functor "if–then".

### [A. Implication]

**9. The principle of identity**

I denote propositional variables by *p*, *q*, *r*, *s*,... , and the functor "if" by *C*. The expression *Cpq* means "if *p*, then *q*" ("then" may be omitted), and is called "implication" with *p* as the antecedent and *q* as the consequent. *C* does not belong to the antecedent, it combines only the antecedent with the consequent. The simplest *C*–thesis is the principle of identity:

(9.1) *Cpp*.

In words: "If *p*, then *p*". Examples: "If today is Tuesday, then today is Tuesday". "If 3 is less than 5, then 3 is less than 5". This thesis is of no use for proving *p*, because the proof would be circular, but it is very important as a premiss. We can derive from it many other theses by substitution. The rule of substitution will be exactly formulated in the systematic part; for the moment it suffices to know that for the propositional variables may be substituted any significant expressions, and in our *C*–theory an expression is called significant, when it is either a variable or an implication having significant expressions as antecedent and consequent. Such an expression is for instance *Cpq*, because its antecedent and consequent, *p* and *q*, are significant expressions. By substituting *Cpq* for *p* we obtain a new thesis:

(9.2) *CCpqCpq*.

In a similar way we can get by substitution the theses *CCCpqrCCpqr*, *CCpCqrCpCqr*, *CCCpqCpqCCpqCpq*, and so on. No new theses can be derived from (9.1) by detachment, although it is possible to apply the rule of detachment to some of its consequences. Take for instance the thesis *CCCpqCpqCCpqCpq*. Its antecedent is the thesis (9.2), so we can detach the consequent; but we cannot get in this way a new thesis, as in all substitutions of the principle of identity the consequent must be identical with the antecedent.

It is obvious that from *Cpp* can be derived *Cqq*, and from *Cqq* again *Cpp*. *Cpp* is equivalent to *Cqq*, which is another form of the principle of identity. From *Cpp* can also be derived *CCpqCpq*, but it is impossible to obtain *Cpp* from *CCpqCpq* by the rules of substitution and detachment. We say that *CCpqCpq* is weaker than *Cpp*, and *Cpp* is stronger than and independent of *CCpqCpq*. This independence must be proved, and the general method of proving independences may be described as follows:

If we have to prove that a thesis *B* of a system is independent of a set of theses *A* of the same system (the set *A* may consist of only one thesis), it is sufficient to find a property φ satisfying the following three conditions:

(a) φ must belong to every thesis of the set $A$;

(b) φ must be hereditary with respect to the rules of inference accepted in the system;

(c) φ must not belong to the thesis $B$.

A property is called hereditary with respect to a rule of inference, if it is transmitted by the rule from the premisses to the conclusion. When the above conditions are satisfied, it is plain that $B$ cannot be a consequence of $A$, since all the consequences of $A$ must have the property φ, which does not belong to $B$.

The ordinary proofs of independence given by the mathematicians, *e.g.* that the axiom of parallels is independent of all the other axioms of the Euclidean geometry, are performed according to a special case of the above general method. They all are called proofs by interpretation, for they are based on an interpretation of constant terms occurring in $A$ and in $B$, which verifies $A$ without verifying $B$. In this case is the property "being true", and it is plain that this property is hereditary with respect to all valid rules of inference.

A property φ can be easily found in our example. Thesis (9.2) begins with two *C*'s, and all its consequences must also begin with two *C*'s, for all its consequences can be obtained by substitution, and [at] the beginning two functors cannot be altered by any substitution. We see therefore, that the structural property: "beginning with two *C*'s", belongs to the thesis *CCpqCpq*, is hereditary with respect to the rules of substitution and detachment, the sole rules of inference accepted in the *C*–system, and does not belong to *Cpp* which begins with one *C*. *Cpp* therefore is independent of *CCpqCpq*. I shall give in the next section another proof of this independence based on the so called matrix–method.

*Cpp* is not only independent of *CCpqCpq*, it is, of course, independent of the whole set of its weaker consequences. This fact is of some interest philosophically. You can sometimes meet the opinion that if all the weaker consequences of a universal law are true, the law itself must be true. The afore–cited example shows that this opinion is not generally true.

### 10. The principle of simplification

The second simplest *C*–thesis is the following one, called in *Principia mathematica* (rather improperly) "the principle of simplification":

(10.1) *CpCqp*.

In words: "If $p$, then if $q$, then $p$"; $p$ is the antecedent, *Cqp* the consequent. This principle is not so evident as the principle of identity, but it can be brought to evidence thus: Let us take for $p$ the true proposition: "3 is less than 5"; we get by detachment: "If $q$, then 3 is less than 5", where $q$ is any proposition whatever. When $q$ means: "Today is Tuesday", we get the true implication: "If today is Tuesday, then 3 is less than 5". When $q$ means: "Today is not Tuesday", we get the true implication: "If today is not Tuesday, then 3 is less than 5". It does not matter, of course, whether today is Tuesday or not, 3 is always less than 5, this proposition is true under any condition. Speaking generally: if $\alpha$ is a true proposition, then the implication $Cq\alpha$ is always true; or in other words: if the consequent of an implication is true, then the implication is true without regard to the antecedent. This principle was known to the mediaeval logicians who have formed the rule: *verum sequitur ad quodlibet*.

The principle of simplification enables us to adjoin to a thesis any antecedent you choose. So we get for instance:

(9.1) *Cpp*

(10.1) *p* / *Cpp* * C(9.1)–(10.2)

(10.2) *CqCpp*

According to the derivational line preceding this thesis, we first obtain by the substitution *p* / *Cpp* the thesis *CCppCqCpp*, which is here omitted for the sake of brevity, and then from this thesis we get by detachment *CqCpp*.

Applying (10.1) to itself we get another example of a thesis with an adjoint antecedent:

(10.1) *p* / *CpCqp*, *q* / *r* * C(10.1)–(10.3)

(10.3) *CrCpCqp*.

In a similar way we can obtain from (10.3) the thesis *CsCrCpCqp* and then *CtCsCrCpCqp*, and so on. All these consequences of the principle of simplification are equivalent to this principle, because from *CrCpCqp*, for instance, we can get again *CpCqp* by substitution *r* / *CrCpCqp* and detachment. There exist also weaker consequences of (10.1), and two of them deserve our attention:

(10.1) *p* / *CpCqp*, *q* / *Crr* * C(10.1)–(10.4)

(10.4) *CCrrCpCqp*

(10.1) *p* / *CpCqp*, *q* / *CpCqp* * C(10.1)–(10.5)

(10.5) *CCpCqpCpCqp*.

It was shown by A. Tarski that each of these consequences is independent of *CpCqp*, but both taken together yield *CpCqp*. The latter proof is easy:

(10.4) *r* / *CpCqp* * C(10.5)–(10.1)

(10.1) *CpCqp*.

The proof that *CpCqp* is independent of (10.4) and of (10.5) taken separately is not so easy. I shall first prove that *CpCqp* is independent of *CCpCqpCpCqp*. The proof is based on a structural property of the thesis (10.5): every variable occurs in this thesis an even number of times. This property, which does not belong to *CpCqp*, is hereditary with respect to the rules of substitution and detachment. This is evident for the rule of substitution, because whatever significant expression we may put for a variable we must do that an even number of times. Further, if every variable occurs an even number of times in $C\alpha\beta$ and $\alpha$, then it must also occur an even number of times in $\beta$. Let the number of occurrences of a variable contained in $\beta$ be $2n$ in $C\alpha\beta$, and $2k$ in $\alpha$; then the number of its occurrences in $\beta$ is $2n-2k = 2(n-k)$, thence it is even. On the basis of this property we can not only prove that *CpCqp* is independent of *CCpCqpCpCqp*, but that is also independent of *Cpp*. Proofs of independence based on the property that every variable occurs an even number of times were first given by Tarski.

It is not always easy to find a structural property needed for a proof of independence. We want to have a general method which could be applied to any case. Such a method was first published by P. Bernays, but at the same time it was invented by myself independently of Bernays. I call it the matrix–method, because it is connected with many–valued systems of logic built up upon the so called "matrices". I shall explain this method by an example.

Let us assume that the figures 1, 2 and 3 denote three constant values of propositional variables. It is irrelevant what is the meaning of those values; we have only to define which value belongs to an implication having 1, 2 or 3 as antecendent and consequent.

This can be done in several ways. For our purpose we accept the following 9 definitions (the sign "=" can be read "means the same as"):

C11 = 1, C12 = 2, C13 = 2,
C21 = 1, C22 = 1, C23 = 1,
C31 = 1, C32 = 2, C33 = 2.

| C | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 2 | 2 |
| 2 | 1 | 1 | 1 |
| 3 | 1 | 2 | 2 |

$M_1$

It is convenient to write down these equalities in form of a table which is called a matrix. The first argument is in the left column, the second one in the top line of the matrix. The purpose of this matrix, and of all similar matrices, is to construct a property φ hereditary with respect to the rules of substitution and detechment. We select for this purpose a value among the figures 1, 2 and 3, for instance 1, and mark it by an asterisk. We say that a thesis is verified by the matix $M_1$, if for all combinations of the values 1, 2 and 3, put in the thesis for the variables and reduced according to the matrix, we obtain the selected value 1. That will be better understood by an example. I shall show that the thesis *CCpqCpq* is verified by the matrix $M_1$. For this purpose we must form the following nine combinations:

$p/1, q/1$: CC11C11 = C11 = 1, $p/2, q/1$: CC21C21 = C11 = 1,
$p/1, q/2$: CC12C12 = C22 = 1, $p/2, q/2$: CC22C22 = C11 = 1,
$p/1, q/3$: CC13C13 = C22 = 1, $p/2, q/3$: CC23C23 = C11 = 1,
$p/3, q/1$: CC31C31 = C11 = 1,
$p/3, q/2$: CC32C32 = C22 = 1,
$p/3, q/3$: CC33C33 = C22 = 1.

I shall explain these formulae for the substitution $p/3, q/2$. We get by this substitution from *CCpqCpq* the expression CC32C32. According to the matrix, C32 means the same as 2, we replace therefore C32 by 2 getting from CC32C32 the expression C22. Now C22 means the same as 1 according to the matrix. So we obtain finally by reduction 1. Since for all combinations of the values 1, 2 and 3 put for the variables we get as the final result 1, we say that *CCpqCpq* is verified by the matrix $M_1$.

The property "to be verified by the matrix $M_1$" is hereditary with respect to the rules of substitution and detachment. This is obvious for the rule of substitution, since the range of the values cannot be enlarged by any substitution. So for instance, thesis *CCpqCpq* is verified by the matrix $M_1$; thence thesis *CCpCqpCpCqp* which follows from *CCpqCpq* by the substitution $q/Cqp$, also must be verified. For *Cqp* must assume one of the values 1, 2, 3, and it was shown that *CCpqCpq* is verified by the matrix for all the values of *p* and *q*. Every thesis verified by a matrix transmits the property of verifiability by this matrix to all its consequences which may be obtained by substitution.

A sufficient, but not a necessary condition, that the property of verifiability by a matrix should be hereditary with respect to the rule of detachment, can be described as follows: Let *i* and *j* be values of a matrix; if *i* is a selected value, then *Cij* has a selected value only when also *j* is a selected value. So for instance in our matrix $M_1$, C11 has the selected value 1, as its antecedent and its consequent are both selected values, but C12 and C13 have the value 2, as their consequents 2 and 3 are not selected values. A matrix which satisfies the above condition is called "normal". Every thesis verified by a normal matrix transmits the property of verifiability by this matrix to all its consequences which may be obtained by detachment. Matrix $M_1$ is normal, and we see at once, that if $C\alpha\beta = 1$ and $\alpha = 1$, then $\beta$ must be 1, since for $\beta = 2$ or 3, $C\alpha\beta$ ($= C1\beta$) cannot have the value 1.

Now, *CCpqCpq* is verified by the matrix $M_1$, but not *CpCqp*, because for $p/3$, $q/3$ we get: C3C33 = C32 = 2. *CpCqp* therefore is independent of *CCpqCpq*, and also of *CCpCqpCpCqp*. It can be proved by the same matrix that *Cpp* is independent of *CCpqCpq*, for *CCpqCpq* is verified by $M_1$, but not *Cpp* which gives for $p/3$ the not selected value 2.

Returning to our problem, set forth on p. 18, we must now prove that *CpCqp* is independent of *CCrrCpCqp*. This prove is based on a structural property. I shall show by a method invented by Tarski that no thesis can be obtained from *CCrrCpCqp* by detachment. I shall call such theses "undetachable". The method of proving that a thesis is undetachable is very simple. If we want to derive from (10.4), i.e. *CCrrCpCqp*, a thesis by detachment, we must get two substitutions of (10.4), one of them being $C\alpha\beta$, and the other $\alpha$. Let us suppose that $C\alpha\beta$ follows from (10.4) by the substitutions $r/\gamma$, $p/\delta$, $q/\varepsilon$, and has the form:

(10.6) $CC\gamma\gamma C\delta C\varepsilon\delta$,

and $\alpha$ results by the substitutions $r/\kappa$, $p/\lambda$, $q/\mu$, having the form:

(10.7) $CC\kappa\kappa C\lambda C\mu\lambda$.

Thesis (10.7), being $\alpha$, must be equiform to the antecedent of (10.6) which is $C\alpha\beta$; we may write:

(10.8) $CC\kappa\kappa C\lambda C\mu\lambda \cong C\gamma\gamma$,

where the geometrical symbol "$\cong$" represents the relation of equiformity. Because the left side of the formula (10.8) is equiform to its right side, it follows that the antecedent of the left side must be equiform to the antecedent of the right side, and the consequent of the left side to the consequent of the right side. We get therefore the formulae:

(10.9) $C\kappa\kappa \cong \gamma$ and (10.10) $C\lambda C\mu\lambda \cong \gamma$.

It follows from these formulae that

(10.11) $C\kappa\kappa \cong C\lambda C\mu\lambda$,

because both $C\kappa\kappa$ and $C\lambda C\mu\lambda$ are equiform to the same expression $\gamma$. Applying to (10.11) the same way of reasoning as to (10.8), we obtain two other formulae:

(10.12) $\kappa \cong \lambda$ and (10.13) $\kappa \cong C\mu\lambda$,

and from these formulae results the final consequence:

(10.14) $\lambda \cong C\mu\lambda$.

This consequence is obviously false, for no expression can be equiform to a part of itself. Our supposition that two substitutions can be got from *CCrrCpCqp*, one of the form $C\alpha\beta$, and the other of the form $\alpha$, leads to a false consequence, therefore cannot be true. Thesis *CCrrCpCqp* is undetachable.

It follows from this that all the consequences of (10.4) are obtainable only by substitution. It is obvious that by substitution we cannot derive *CpCqp* from *CCrrCpCqp*. *CpCqp* therefore is independent of *CCrrCpCqp*.

Our problem concerning the principle of simplification is now solved completely. *CpCqp* is equivalent to its two weaker consequences, *CCrrCpCqp* and *CCpCqpCpCqp*. We say that it is "decomposable" into two weaker theses. It was shown by Tarski that there are only a few theses which are decomposable in this manner.

**11. The principle of the syllogism**

The most powerful and most frequently used instrument of proof is the principle of the hypothetical syllogism. It runs:

(11.1) *CCpqCCqrCpr*.

In words: "If (if *p*, then *q*), then if (if *q*, then *r*), then (if *p*, then *r*)". The construction of this formula was explained in the "Introduction" p. 9. To make this principle evident I shall give an example. Let *p* mean: "today is the first Monday in August"; *q* – "today is a Bank Holiday"; *r* – "the banks are legally closed". From the implications: "if today is the first Monday in August, then today is a Bank Holiday", and "if today is a Bank Holiday, then the banks are legally closed", there follows according to the principle of the syllogism that "if today is the first Monday in August, then the banks are legally closed".

I shall now derive from this principle some consequences which also are important.

(11.1) *p*/*Cpq*, *q*/*CCqrCpr*, *r*/*s* * C(11.1)–(11.2)

(11.2) *CCCCqrCprsCCpqs*

(11.2) *p*/*s*, *s*/*CpCsr* * (11.3)

(11.3) *CCCCqrCsrCpCsrCCsqCpCsr*

(11.2) *q*/*Cqr*, *r*/*Csr*, *s*/*CCsqCpCsr* * C(11.3)–(11.4)

(11.4) *CCpCqrCCsqCpCsr*

(11.4) *p*/*CpCqr*, *q*/*Csq*, *r*/*CpCsr*, *s*/*t* * C(11.4)–(11.5)

(11.5) *CCtCsqCCpCqrCtCpCsr*

(11.5) *t*/*Cpq*, *s*/*Cqr*, *q*/*Cpr*, *p*/*s*, *r*/*t* * C(11.1)–(11.6)

(11.6) *CCsCCprtCCpqCsCCqrt*

(11.6) *s*/*Cpq*, *p*/*q*, *t*/*Cpr*, *q*/*s* * C(11.1)–(11.7)

(11.7) *CCqsCCpqCCsrCpr*

Among these consequences the most important are theses (11.4) and (11.7). The latter thesis is a kind of *sorites* with a twist in the premisses, as the natural order of them would be *CCpqCCqsCCsrCpr*. Theses *CCpqCCqrCpr* and *CCpqCCqsCCsrCpr* are independent of each other (try to prove this by the matrix method). We can obtain the principle of the syllogism from the *sorites* *CCpqCCqsCCsrCpr* by means of the principle of identity, thus:

(9.1) *Cpp*

(11.8) *CCpqCCqsCCsrCpr*

(11.8) *q*/*p*, *s*/*q* * C(9.1)–(11.1)

(11.1) *CCpqCCqrCpr*,

but matrix $M_1$ shows that *Cpp* is independent of *CCpqCCqrCpr* and of the *sorites* (11.8), which both are verified by this matrix. All the three principles, *Cpp*, *CpCqp* and *CCpqCCqrCpr* are independent of each other.

There exists a second form of the principle of the syllogism of an almost equal importance:

(11.9) *CCqrCCpqCpr*.

The difference between (11.1) *CCpqCCqrCpr* and (11.9) *CCqrCCpqCpr* can be described in the following way: The first principle is of the form C1C23, whereas the second has the form C2C13. 3 means here the conclusion *Cpr*, 1 and 2 are the premisses *Cpq* and *Cqr* upon which the conclusion depends. We see that the premisses are commuted in (11.9) with regard to (11.1). This is irrelevant, when both premisses are true, but it makes a difference, when neither of them is true, or only one. All the consequences of the second form of the syllogism are different from those of the first form. Let us deduce some of them:

(11.9) *q* / *Cqr*, *r* / *CCpqCpr*, *p* / *s* * C(11.9)–(11.10)

(11.10) *CCsCqrCsCCpqCpr*

(11.10) *s* / *Cqr*, *q* / *Cpq*, *r* / *Cpr*, *p* / *s* * C(11.9)–(11.11)

(11.11) *CCqrCCsCpqCsCpr*

(11.11) *q* / *Cqr*, *r* / *CCpqCpr*, *p* / *t* * C(11.9)–(11.12)

(11.12) *CCsCtCqrCsCtCCpqCpr*

(11.12) *s* / *Cqr*, *t* / *Cpq*, *q* / *p*, *p* / *s* * C(11.9)–(11.13)

(11.13) *CCqrCCpqCCspCpr*

The last consequence is also a kind of *sorites* with another twist in the premisses. Neither form of the syllogism gives the ordinary *sorites* as stated in (11.8).

The two forms of the syllogism are independent of each other. That the first form is independent of the second, can be proved by the matrix $M_2$ which verifies (11.9) without verifying (11.1), because we get from *CCpqCCqrCpr* by the substitution *p*/2, *q*/1, *r*/3: CC21CC13C23 = C1C23 = 2. The matrix proving that *CCqrCCpqCpr* is independent of *CCpqCCqrCpr* is more complicated (try to find it).

| C | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 2 | 2 |
| 2 | 1 | 1 | 3 |
| 3 | 1 | 1 | 1 |

$M_2$

The principle of the hypothetical syllogism was known to Aristotle and was studied by Theophrastus. It is one of a few theses of the propositional logic known to mathematicians and employed by them.

### 12. The principle of commutation

Another very important C–thesis is the so called "principle of commutation". It runs thus:

(12.1) *CCpCqrCqCpr.*

In words: "If [if *p*, then (if *q*, then *r*)], then [if *q*, then (if *p*, then *r*)]". The antecedent of this formula is *CpCqr*, the consequent *CqCpr*. We see that *p* and *q* are commuted in the consequent with regard to the antecedent. Example: Let us take as antecedent of (12.1) the proposition: "if *N* is an even number, then if it is divisible by 3, it is divisible by 6"; then the consequent will be: "if *N* is divisible by 3, then if it is even, it is divisible by 6". It is obvious that this consequent must be true, provided the antecedent is true.

By the principle of commutation we can get at once from the first form of the syllogism its second form, and vice versa. Here are the respective deductions:

From *CCpqCCqrCpr* to *CCqrCCpqCpr*;

(12.1) *p* / *Cpq*, *q* / *Cqr*, *r* / *Cpr* * C(11.1)–(11.9)

(11.9) *CCqrCCpqCpr*

From *CCqrCpqCpr* to *CCpqCCqrCpr*:

(12.1) *p* / *Cqr*, *q* / *Cpq*, *r* / *Cpr* * C(11.9)–(11.1)

(11.1) *CCpqCCqrCpr*.

Applying (12.1) to itself we get the following consequence:

(12.1) $p / CpCqr, r / Cpr$ * C(12.1)–(12.2)

(12.2) *CqCCpCqrCpr*.

This consequence is equivalent to the principle of commutation, because *CCpCqrCqCpr* follows from *CqCCpCqrCpr*:

(12.2) $q / CqCCpCqrCpr, p / s, r / t$ * C(12.2)–(12.3)

(12.3) *CCsCCqCCpCqrCprtCst*

(12.2) $q / CpCqr, p / q, r / Cpr$ * (12.4)

(12.4) *CCpCqrCCqCCpCqrCprCqCpr*

(12.3) $s / CpCqr, t / CqCpr$ * C(12.4)–(12.1)

(12.1) *CCpCqrCqCpr*.

This neat deduction was given by Tarski.

There exist many interesting connexions between the principle of commutation and the foregoing three principles. From the principle of commutation together with *CCpqCpq*, a consequent of the principle of identity results the following thesis:

(9.2) *CCpqCpq*

(12.1) $p / Cpq, q / p, r / q$ * C(9.2)–(12.5)

(12.5) *CpCCpqq*.

In words: "If *p*, then if (if *p*, then *q*), then *q*". A thesis equivalent to (12.5), but formulated with "and": "if (*p* and if *p*, then *q*), then *q*", is called in *Principia mathematica* "the principle of assertion", and is explained (rather loosely) as follows: "if *p* is true, and *q* follows from it, then *q* is true". I shall retain the name "principle of assertion" for thesis (12.5).

The principle of assertion gives with either form of the syllogism the principle of commutation. Thesis (12.1) results directly from *CqCCqrr*, another form of (12.5), and (11.4), a consequence of the first form of the syllogism, and indirectly by means of (12.2) from *CqCCqrr* and (11.10), a consequence of the second form of the syllogism. The deductions proceed as follows:

(12.5) $p / q, q / r$ * (12.6)

(12.6) *CqCCqrr*

(11.4) *CCpCqrCCsqCpCsr*

(11.4) $p / q, q / Cqr, s / p$ * C(12.6)–(12.1)

(12.1) *CCpCqrCqCpr*

*

(11.10) *CCsCqrCsCCpqCpr*

(11.10) $s / q, q / Cqr$ * C(12.6)–(12.2)

(12.2) *CqCCpCqrCpr*.

Thesis (12.2) is equivalent to the principle of commutation.

The principle of commutation is independent of all the former principles taken separately.

We may remark at the end that from the principle of simplification taken together with the principle of commutation follows the principle of identity:

(10.1) *CpCqp*

(12.1) $r / p$ * C(10.1)–(10.2)

(10.2) *CqCpp*

(10.2) $q / CqCpp$ * C(10.2)–(9.1)

(9.1) *Cpp*.

**13. The principle of Frege**

There exists a C–thesis not much important as to its own consequences, but very powerful when combined with some other theses. It is the following one:

(13.1) *CCpCqrCCpqCpr.*

In words: "If if *p*, then (if *q*, then *r*), then if (if *p*, then *q*), then (if *p*, then *r*)". Its antecedent is *CpCqr*, its consequent *CCpqCpr*. This thesis resembles on the one side to the second form of the syllogism, and on the other side to the principle of commutation. If you drop *Cp* in the antecedent, you will get *CCqrCCpqCpr*, and if you drop *Cp* in the consequent, you will get *CCpCqrCqCpr*. The sense of (13.1) may be explained as follows: The antecedent *CpCqr* asserts that *r* depends upon two conditions, *p* and *q*; the consequent *CCpqCpr* states that if the second condition *q* depends on the first *p*, then *r* depends only on *p*. Example: Let *p* mean: "the sum of the figures of number *N* is divisible by 9"; *q* — "*N* is divisible by 9"; *r* — "*N* is divisible by 3". From *CpCqr* we get the implication: "if the sum of the figures of number *N* is divisible by 9, then if *N* is divisible by 9, *N* is divisible by 3". But *Cpq* is true, for "if the sum of the figures of number *N* is divisible by 9, then *N* is divisible by 9". Therefore *Cpr* is true: "if the sum of the figures of number *N* is divisible by 9, then *N* is divisible by 3".

I have called (13.1) "the principle of Frege", because Frege discovered this thesis a[nd] raised it to the dignity of an axiom together with some theses with negation and two other C–theses, the principle of simplification and the principle of commutation. Frege's system of axioms, however, is not independent. It can be shown that the principle of commutation follows from thesis (13.1) and the principle of simplification. The deduction is somewhat long, but instructive.

The premisses are:

(10.1) *CpCqp*

(13.1) *CCpCqrCCpqCpr*

It follows from these premisses:

(10.1) *p* / *CCpCqrCCpqCpr*, *q* / *s* * C(13.1)–(13.2)

(13.2) *CsCCpCqrCCpqCpr*

(13.1) *p* / *s*, *q* / *CpCqr*, *r* / *CCpqCpr* * C(13.2)–(13.3)

(13.3) *CCsCpCqrCsCCpqCpr*

(10.1) *p* / *Cqr*, *q* / *p* * (13.4)

(13.4) *CCqrCpCqr*

(13.3) *s* / *Cqr* * C(13.4)–(13.5)

(13.5) *CCqrCCpqCpr*

(13.1) *p* / *Cqr*, *q* / *Cpq*, *r* / *Cpr* * C(13.5)–(13.6)

(13.6) *CCCqrCpqCCqrCpr*

(10.1) *p* / *CpCqp*, *q* / *CCqpr* * C(10.1)–(13.7)

(13.7) *CCCqprCpCqp*

(13.6) *q* / *Cqp* * C(13.7)–(13.8)

(13.8) *CCCqprCpr*

(13.8) *q* / *p*, *p* / *q*, *r* / *Cpr* * (13.9)

(13.9) *CCCpqCprCqCpr*

(13.5) *q* / *CCpqCpr*, *r* / *CqCpr*, *p* / *s* * C(13.9)–(13.10)

(13.10) *CCsCCpqCprCsCqCpr*

(13.10) *s* / *CpCqr* * C(13.1)–(12.1)

(12.1) *CCpCqrCqCpr*.

It can be easily seen that from Frege's principle, taken together with the principle of simplification, result all the theses which were hitherto established. Thesis (13.5) is the second form of the principle of the syllogism, and gives together with commutation the first form of the syllogism. From simplification and commutation follows identity, and from identity and commutation the principle of assertion.

There is another important consequence of theses (10.1) and (13.1) which directly follows from the principles of assertion and Frege:

(12.5) *CpCCpqq*

(13.1) *CCpCqrCCpqCpr*

(13.1) *q* / *Cpq*, *r* / *q* * C(12.5)–(13.11)

(13.11) *CCpCpqCpq*.

Thesis (13.11) means in words: "If if *p*, then (if *p*, then *q*), then (if *p*, then *q*)". *CpCpq* is the antecedent, and *Cpq* the consequent. The antecedent asserts that *q* depends on two conditions which are identical; the consequent states that it suffices to mention this conditions only once. This thesis was known to the Stoics, but was misunderstood by their commentators. The Stoic example was: "If it is day, then if it is day, it is light; but it is day: therefore it is light". To prove this syllogism the Stoics applied to it the *modus ponens* twice: first, from the implication "if it is day, then if it is day, it is light", and the premiss "it is day", they derived by the *modus ponens* the consequence "if it is day, then it is light"; secondly, from the implication "if it is day, it is light", and the premiss "it is day" they derived again by the *modus ponens* the conclusion "it is light".

Thesis *CCpCpqCpq* yields assertion and commutation, when it is combined with simplification and the first form of the syllogism, according to the following deduction:

(10.1) *CpCqp*

(11.1) *CCpqCCqrCpr*

(13.11) *CCpCpqCpq*

(11.2) *CCCCqrCprsCCpqs* (first consequence of (11.1))

(13.11) *p* / *Cqr*, *q* / *r* * (13.12)

(13.12) *CCCqrCCqrrCCqrr*

(11.2) *p* / *Cqr*, *s* / *CCqrr* * C(13.12)–(13.13)

(13.13) *CCCqrqCCqrr*

(11.1) *q* / *Cqp* * C(10.1)–(13.14)

(13.14) *CCCqprCpr*

(13.14) *q* / *Cqr*, *p* / *q*, *r* / *CCqrr* * C(13.14)–(12.6)

(12.6) *CqCCqrr*.

(12.6) is the principle of assertion; from (12.6) and (11.1) results the principle of commutation, as was shown above.

Thesis *CCpCpqCpq* may be called "the principle of Hilbert", since D. Hilbert included it into the axioms of his system of logic. Both principles, that of Frege and that of Hilbert, are independent of all the foregoing principles. It is easy to check that matrix $M_3$ verifies identity, simplification, both forms of the syllogism, commutation and assertion. Frege's principle is not verified, for we get for *p* / 2, *q* / 2, *r* / 3: CC2C23CC22C23 = CC22C12 = C1C12 = C12 = 2, nor Hilbert's principle is verified, because we have for *p* / 2, *q* / 3: CC2C23C23 = CC222 = C12 = 2.

| C | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 2 | 1 |
| 2 | 1 | 1 | 2 |
| 3 | 1 | 1 | 1 |

$M_3$

**14. The so–called "positive logic"**

The system of consequences which can be deduced by the rules of substitution and detachment from the principle of simplification and the principle of Frege, i.e. from:

(10.1) *CpCqp*

(13.1) *CCpCqrCCpqCpr*,

forms a natural part of the implicational system, i.e. of the system of C–theses, and was called by Bernays "positive logic". In two system[s] of theory of deduction, that of Frege and that of Hilbert, the axioms without negation are positive logic, and other C–theses not belonging to this logic can be derived only by the help of axioms with negation. In the so called "intuitionist" logic, founded by Brouwer and Heyting, only those C–theses are accepted that belong to positive logic.

The set of C–axioms in Frege's system consists of three theses, simplification, Frege's principle, and commutation, and is not independent, for commutation follows from simplification and Frege's principle. The set of C–axioms in Hilbert's system consists of the following four theses:

(10.1) *CpCqp*

(11.9) *CCqrCCpqCpr*

(12.1) *CCpCqrCqCpr*

(13.11) *CCpCpqCpq*.

The set of these four theses is equivalent to the theses *CpCqp* and *CCpCqrCCpqCpr* of Frege's system. The proof in one direction, from Frege to Hilbert, was given above; in the other direction, from Hilbert to Frege, we must only prove that *CCpCqrCCpqCpr* results from the theses of Hilbert, as *CpCqp* is common to both sets of theses. This proof runs as follows:

As premisses we take three axioms of Hilbert, (11.9), (12.1) and (13.11), as (10.1) is not needed for the proof. The conclu[sion] will be thesis (13.1).

(11.11) *CCqrCCsCpqCsCpr* (a consequence of (11.9))

(11.11) *q* / *CpCpq*, *r* / *Cpq*, *p* / *t* * C(13.11)–(14.1)

(14.1) *CCsCtCpCpqCsCtCpq*

(11.9) *r* / *Cpr* * (14.2)

(14.2) *CCqCprCCpqCpCpr*

(14.1) *s* / *CqCpr*, *t* / *Cpq*, *q* / *r* * C(14.2)–(14.3)

(14.3) *CCqCprCCpqCpr*

(11.9) *q* / *CqCpr*, *r* / *CCpqCpr*, *p* / *s* * C(14.3)–(14.4)

(14.4) *CCsCqCprCsCCpqCpr*

(14.4) *s* / *CpCqr* * C(12.1)–(13.1)

(13.1) *CCpCqrCCpqCpr*.

The set of the four axioms of Hilbert is independent. Matrix $M_1$ proves the independence of simplification of the remaining axioms, $M_2$ that of commutation, and $M_3$ that of the principle of Hilbert. In order to prove that the second form of the syllogism is independent of all the other axioms, we must have a new matrix, $M_4$. Neither form of the syllogism is verified by this matrix because we get for $q/2$, $r/3$, $p/1$: $CC23CC12C13 = C1C13 = C13 = 3$, whereas all the other axioms are verified. Matrix $M_4$ is not normal, for $C12 = 1$, nevertheless the property to be verified by it is hereditary with respect to the rule of detachment. It is easy to see that according to this matrix the implication *Cpq* can never assume the value 2; $C1\beta$ therefore can have the value 2 in the consequent only when $\beta$ is a variable. But in this case $C1\beta = C1q$, and cannot be 1, because we can put 3 for $q$ getting thus $C13 = 3$. If therefore $C\alpha\beta = 1$ and $\alpha = 1$, then $\beta$ must be 1.

| C | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 1 | 3 |
| 2 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 |

$M_4$

If we replace in Hilbert's set of axioms the second form of the syllogism by the first form, we can drop the principle of commutation which follows from the remaining axioms, as was shown in the foregoing section. The set of axioms:

(10.1) *CpCqp*

(11.1) *CCpqCCqrCpr*

(13.11) *CCpCpqCpq*

is a base of positive logic, like theses (10.1) and (12.1).

**15. The principle of Peirce**

There is a lot of C–theses independent of positive logic, and one of the strongest among them is the following thesis:

(15.1) *CCCpqpp*.

*CCpqp* is the antecedent, *p* is the consequent. It would be in vain to try to explain this th[e]sis in words or to make it evident by the help of other C–theses. For the moment it suffices to observe that *p* is either true or false. If *p* is true, then thesis (15.1) is true according to the scholastic principle: *verum sequitur ad quodlibet*. If *p* is false, then according to another scholastic principle: *ad falsum sequitur quodlibet*, i.e. a false proposition entails any proposition, *Cpq* is true, and therefore *CCpqp* is false, because its antecedent is true and its consequent false. Thence thesis (15.1) is true according to the second scholastic principle.

Thesis (15.1) I have called "the principle of Peirce" for C.S. Peirce first discovered this thesis. It does not occur either in Frege's work or in *Principia mathematica*. It is independent of all the foregoing theses. Matrix $M_5$ verifies the positive logic without verifying Peirce's principle, because we get for $p/2$, $q/4$: $CCC2422 = CC422 = C12 = 2$. The principle of Peirce gives together with simplification and the first form of the syllogism the whole implicational logic, as we shall see in the next section.

| C | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *1 | 1 | 2 | 3 | 4 |
| 2 | 1 | 1 | 3 | 4 |
| 3 | 1 | 2 | 1 | 4 |
| 4 | 1 | 1 | 3 | 1 |

$M_5$

Another thesis, independent of positive logic, reads as follows:

(15.2) *CCCpqqCCqpp.*

*CCpqq* is the antecedent, *CCqpp* the consequent. It is as difficult to explain the meaning of this thesis in words, as to explain the meaning of Peirce's principle. But, as we shall see later, *CCpqq* may be taken as *definiens* of the alternation, i.e. of the function "*p* or *q*", in symbols *Apq;* then *CCqpp* will mean *Aqp,* and thesis (15.2) will be equivalent to the commutative principle of alternation: *CApqAqp,* which is evident.

Peirce's principle easily follows from (15.2) and the principle of Hilbert:

(13.11) *CCpCpqCpq*

(15.2) *CCCpqqCCqpp*

(15.2) *q / Cpq* * C(13.11)–(15.1)

(15.1) *CCCpqpp.*

Knowing already that Peirce's principle is independent of positive logic, we can prove the independence of (15.2) without recurring to a matrix: since (15.1) is independent of positive logic, so must be also (15.2), from which follows (15.1) in connexion of a thesis of this logic.

The next thesis:

(15.3) *CCCpqrCCrpp*

can be regarded as a generalization of the former thesis, for we get from it (15.2) by the substitution *r / q*. Of a similar kind is the thesis:

(15.4) *CCCpqrCCprr,*

which resembles to the so called "constructive dilemma". *CCpqr* is the antecedent, *CCprr* the consequent. The thesis assert that if *r* follows from *Cpq* and from *p*, and both premisses *Cpq* and *p* are true, then *r* is true. From (15.4) we get Peirce's principle by commutation and identity:

(9.1) *Cpp*

(12.1) *CCpCqrCqCpr*

(15.4) *CCCpqrCCprr*

(12.1) *p / CCpqr, q / Cpr* * C(15.4)–(15.5)

(15.5) *CCprCCCpqrr*

(15.5) *r / p* * C(9.1)–(15.1)

(15.1) *CCCpqpp.*

There are theses which are independent of the positive logic, but weaker than the principle of Peirce. Of such a kind is the following thesis:

(15.6) *CCCpqrCCCqprr.*

*CCpqr* is the antecedent, *CCCqprr* the consequent. The thesis is a constructive dilemma, like (15.4). If *r* follows from *Cpq* and if it follows from *Cqp,* and both premisses, *Cpq* and *Cqp,* are true, then *r* is true. Thesis (15.6) is weaker than Peirce's principle, as can be seen by the matrix $M_5$ which verifies not only the whole positive logic, but also

(15.6), without verifying the principle of Peirce. Another matrix, $M_6$, shows that it is independent from the positive logic. $M_6$ contains $M_5$, but adds to it a line and a column with the new figure 5. The positive logic is again verified, but neither Peirce's principle nor thesis (15.6), because we get for *p*/5, *q*/4, *r*/2: CCC542CCC4522 = CC42CC522 = C1C12 = C12 = 2.

| C | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| *1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 3 | 4 | 5 |
| 3 | 1 | 2 | 1 | 4 | 5 |
| 4 | 1 | 1 | 3 | 1 | 5 |
| 5 | 1 | 1 | 1 | 4 | 1 |

$M_6$

The principle of Peirce is an "undetachable" thesis. This can be proved not only by the method described in section 10, but also by the matrix–method. Matrix $M_7$ verifies this principle. From *CCCpqpp* we have for *p*/1: CCC1*q*11. But C1*q* can be either 2 or 4: we get therefore CC211 or CC411, and in both cases the result is C31 = 1. For *p*/2 we have CCC2*q*22 which gives either CC322 or CC122; in both cases the result is C22 = 1. For *p*/3 we have CCC3*q*33, and we get either CC133 = C43 = 1, or CC233 = C33 = 1. For *p*/4 we have CCC4*q*44, and we get either CC344 = C24 = 1, or CC144 = C44 = 1. $M_7$ has this particularity that *Cpq* does not assume the selected value 1, when its antecedent is 1. This shows that a thesis verified by this matrix cannot give a consequence by detachment, since it is impossible to derive from it two theses of the form $C\alpha\beta = 1$ and $\alpha = 1$, as $C1\beta$ can never have the selected value 1. I do not know any thesis which would be verified by $M_7$, except *CCCpqpp* and its consequences obtained by substitution.

| C | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *1 | 2 | 2 | 4 | 4 |
| 2 | 3 | 1 | 3 | 1 |
| 3 | 1 | 2 | 1 | 2 |
| 4 | 3 | 3 | 1 | 1 |

$M_7$

**16. Implicational logic**

The system containing all C–theses is called "implicational logic". It can be based, as was proved by Tarski, on the following three axioms:

(10.1) *CpCqp*
(11.1) *CCpqCCqrCpr*
(15.4) *CCCpqrCCprr*.

Bernays observed that the third axiom may be replaced by the simpler thesis *CCCpqpp*, and that consequently the set of axioms:

(10.1) *CpCqp* (simplification)
(11.1) *CCpqCCqrCpr* (first form of the syllogism)
(15.1) *CCCpqpp* (Peirce's principle)

forms a sufficient basis for the implicational logic. These three axioms are known as the "Tarski–Bernays" set. The set is independent, i.e. each axiom is independent of the remaining two. For *CCCpqpp* the proof is given by the matrix $M_5$, for *CCpqCCqrCpr* by the matrix $M_4$, and for *CpCqp* by a new matrix $M_8$, which verifies both forms of the syllogism and Peirce's principle without verifying *CpCqp*, as we get for *p*/3 and any *q*: C3C*q*3 = 2.

| C | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *1 | 1 | 2 | 4 | 4 |
| 2 | 1 | 1 | 4 | 4 |
| 3 | 2 | 2 | 1 | 2 |
| 4 | 1 | 2 | 1 | 1 |

$M_8$

The proof that all implicational theses follow from the Tarski–Bernays set by substitution and detachment will be given in the systematic part, where I shall show that this set of axioms is equivalent to the axiom:

(16.1) *CCCpqrCCrpCsp*,

the shortest of all single axioms that suffice to build up the implicational logic. At present I shall deal with some methodically interesting problems connected with the set Tarski–Bernays.

M. Wajsberg has proved that the first axiom of this set, *CpCqp*, may be replaced by any one of the following theses: *CpCCpqq*, *CpCCqpp*, *CpCCqqp*, *CqCpp*, *CqCCCpppp*, *CqCCCpppCCppp*, *CqCCppCpp*, *CqCCppCCppCpp*, *CpCCppp*, *CpCpp*. These results were partially generalized by the author who stated, that *CpCqp* may be replaced by any thesis of the form *CpC*α*p*, provided α is a consequence of the new set of axioms, or by any thesis of the form *Cq*α, where α does not contain variables equiform to *q*. The author has got these results by a kind of inductive argument.

I shall prove below only by substitution and detachment that the axiom *CpCqp* may be replaced in the Tarski–Bernays set by any thesis which has a variable as its antecedent, and an implication as its consequent, i.e. by any thesis of the form *CpC*αβ, where α and β are a new kind of propositional variable. I shall explain this kind of variable after the deduction of *CpCqp* from *CpC*αβ which runs, as follows:

The premisses are:

(16.2) *CpC*αβ

(11.1) *CCpqCCqrCpr*

(15.1) *CCCpqpp*.

It follows from the premisses:

(11.1) *p*/*Cpq*, *q*/*CCqrCpr*, *r*/*s* * C(11.1)–(16.3)

(16.3) *CCCCqrCprsCCpqs* (11.2)

(16.3) *p*/*s*, *s*/*CpCsr* * (16.4)

(16.4) *CCCCqrCsrCpCsrCCsqCpCsr* (11.3)

(16.3) *q*/*Cqr*, *r*/*Csr*, *s*/*CCsqCpCsr* * C(16.4)–(16.5)

(16.5) *CCpCqrCCsqCpCsr* (11.3)

(16.3) *q*/*Cpr*, *r*/*s*, *p*/*Cqr*, *s*/*t* * (16.6)

(16.6) *CCCCCprsCCqrstCCCqrCprt*

(16.5) *p*/*CCCCprsCCqrst*, *q*/*CCqrCpr*, *r*/*t*, *s*/*Cpq* * C(16.6)–C(11.1)–(16.7)

(16.7) *CCCCCprsCCqrstCCpqt*

(16.3) *s*/*CCsrCpr* * (16.8)

(16.8) *CCCCqrCprCCsrCprCCpqCCsrCpr*

(16.7) *p*/*q*, *s*/*Cpr*, *q*/*s*, *t*/*CCpqCCsrCpr* * C(16.8)–(16.9)

(16.9) CCqsCCpqCCsrCpr (11.7)
(16.5) p / Cqs, q / Cpq, r / CCsrCpr, s / t * C(16.9)–(16.10)
(16.10) CCtCpqCCqsCtCCsrCpr
(16.9) q / CCpqp, s / p, p / s * C(15.1)–(16.11)
(16.11) CCsCCpqpCCprCsr
(16.9) q / CsCCpqp, s / CCprCsr, p / t, t / v * C(16.11)–(16.12)
(16.12) CCtCsCCpqpCCCCprCsrvCtv
(16.9) q / α, s / β, p / Csq, r / s * (16.13)
(16.13) CCαβCCCsqαCCβsCCsqs
(11.1) q / Cαβ, r / CCCsqαCCβsCCsqs * C(16.2)–C(16.13)–(16.14)
(16.14) CpCCCsqαCCβsCCsqs
(11.1) p / CCsqα, q / CCβsCCsqs, r / Csq * (16.15)
(16.15) CCCCsqαCCβsCCsqsCCCCβsCCsqsCsqCCCsqαCsq
(11.1) q / CCCsqαCCβsCCsqs, r / CCCCβsCCsqsCsqCCCsqαCsq * C(16.14)–C(16.15)–(16.16)
(16.16) CpCCCCβsCCsqsCsqCCCsqαCsq
(16.5) p / CCsqq, q / CCCβsCCsqsCsq, r / q * (16.17)
(16.17) CCCCsqqCCCCβsCCsqsCsqqCCsCCCβsCCsqsCsqCCCsqqCsq
(16.12) t / p, s / CCCβsCCsqsCsq, p / Csq, q / α, r / q, v / CCsCCβsCCsqsCsqCCCsqqCsq * C(16.16)–C(16.17)–(16.18)
(16.18) CpCCsCCCβsCCsqsCsqCCCsqqCsq
(16.11) s / CCsqs, p / s, q / CCCβsCCsqsCsq * (16.19)
(16.19) CCCCsqsCCsCCCβsCCsqsCsqsCCsrCCCsqsr
(16.12) t / p, s / CsCCCβsCCsqsCsq, p / Csq, r / s, v / CCsrCCCsqsr * C(16.18)–C(16.19)–(16.20)
(16.20) CpCCsrCCCsqsr
(16.10) t / p, p / Csr, q / CCCsqsr, s / CCrsr * C(16.20)–(16.21)
(16.21) CCCCCsqsrCCrsrCpCCCCrsrrCCsrr
(16.7) p / Csq, r / s, s / r, q / r, t / CpCCCCrsrrCCsrr * C(16.21)–(16.22)
(16.22) CCCsqrCpCCCCrsrrCCsrr
(16.22) s / p, q / CCCCrprrCCprr * (16.23)
(16.23) CCCpCCCCrprrCCprrrCpCCCCrprrCCprr
(15.1) p / CpCCCCrprrCCprr, q / r * C(16.23)–(16.24)
(16.24) CpCCCCrprrCCprr
(15.1) p / r, q / CCCpqpp * (16.25)
(16.25) CCCrCCCpqpprr
(16.24) p / CCCpqpp * C(15.1)–C(16.25)–(16.26)
(16.26) CCCCCpqpprr
(16.26) q / p, r / Cqp * (16.27)
(16.27) CCCCCppppCqpCqp
(16.22) s / p, q / p, r / p, p / q * (16.28)
(16.28) CCCpppCqCCCCppppCCppp
(16.12) t / CCppp, s / q, p / CCppp, q / p, r / p, v / Cqp * C(16.28)–C(16.27)–(16.29)
(16.29) CCCpppCqp
(16.24) r / p * (16.30)

(16.30) *CpCCCCppppCCppp*

(16.11) $s/p$, $p/CCppp$, $q/p$, $r/Cqp$ * C(16.30)–C(16.29)–(16.31)

(16.31) *CpCqp*.

If we replace in this deduction the letters α and β in all places where they occur, in the theses as well as in the derivational lines, by such expressions that *CpC*αβ becomes a thesis, we obtain the proof of *CpCpq* based on this thesis and on the principles of syllogism and Peirce. Take *p* for α and *p* for β; then *CpC*αβ becomes *CpCpp*, therefore a thesis, and we get from this thesis and the principles of syllogism and Peirce *CpCqp*. In the same way all the other interpretations of α and β which verify *CpC*αβ, give a proof of *CpCqp*. No interpretation of the Greek letters breaks the coherence of the proof. This proof is a generalization of all the particular results obtained by Wajsberg.

Matrix $M_8$ verifies the principles of the syllogism and of Peirce without verifying *CpC*αβ. Whatever expressions may be denoted by α, and β, we always get C3Cαβ = 2, because Cαβ never assumes the value 3. As $M_8$ also verifies the principle of identity, we see that the set of axioms: *Cpp*, *CCpqCCqrCpr* and *CCCpqpp*, cannot prove the principle of simplification, and consequently cannot form a basis of the implicational logic. There is no possibility to replace the axiom *CpC*αβ by the more general axiom *Cp*α. It is worth–while to observe that, if we accept 2 instead of 1 as value for the implication C33 in $M_8$, the new matrix again verifies the principles of the syllogism and of Peirce, but does not verify any more the principle of identity. No implication the antecedent of which is a variable is verified by this matrix.

### 17. On two kinds of propositional variable

In the proof of the foregoing section I employed Latin as well as Greek letters to denote propositional variables. Although some other authors, among them Wajsberg, also employ Latin and Greek variables, there exists, as far as I know, no publication clearly defining the difference between them. This difference, however, seems to me methodically important.

Latin variables, like *p* or *q*, are connected with a range of values which may be substituted for them. Any significant propositional expression may be a value of *p*, and may be substituted for *p*. I would call them "substitution–variables". Greek variables, like α or β, are also connected with a range of values which are propositional expressions, but during a deduction in which they occur nothing can be substituted for them. Only afterwards they may be interpreted in various ways. I would call them "interpretation–variables". Let us explain by examples the nature and the effects of this essential difference.

If in the above proof based on the axiom *CpC*αβ the Greek variables α and β were replaced by the Latin variables *s* and *t*, the proof would be useless. From the expression:

(I) *CpCst*

we could get at once a variable as consequence, by substitution and two detachments:

(I) $p/CpCst$, $s/CpCst$, $t/p$ * C(I)–C(I)–(II)

(II) *p*,

and from a variable follows by substitution any propositional expression, not only *CpCqp*, but also *CCpqCCqrCpr* and *CCCpqpp*. In this case there would be needless to take pains of performing the above complicated deduction. This deduction has a sense only when α and β represent such expressions that verify *CpC*αβ, i.e. when *CpC*αβ is a thesis. Now, there are various pairs of values for $\alpha$ and $\beta$ that verify *CpC*$\alpha\beta$, e.g.: *p* and *p*, *q* and *q*, *Cpq* and *q*, *CqCpr* and *Cqr*, as any one of the expressions: *CpCpp*, *CpCqq*, *CpCCpqq*, *CpCCqCprCqr*, is a thesis. If we want to have a general formula, we must employ variables, but of another kind than hitherto used. I chose Greek letters for these new variables, but a rule of substitution cannot be stated for them, as it is impossible to express by a general formula the properties that must have $\alpha$ together with $\beta$ in order to verify *CpC*$\alpha\beta$. We only can say: if you find such $\alpha$'s and $\beta$'s that verify *CpC*$\alpha\beta$, the proof based on this axiom is for them valid. In other words, we can nothing substitute for the Greek variables, but we can have various interpretations of them.

Another difference between Latin and Greek variables is made clear by the same example: the Latin variables are unrestricted, as they represent any significant propositional expression, whereas the range of values of the Greek variables is always restricted. They always must satisfy some conditions, in our case the condition that *CpC*$\alpha\beta$ should be a thesis. This fact may be explained by a much simpler example. The usual formulation of the rule of detachment reads: "if *C*$\alpha\beta$ is asserted, and $\alpha$ is asserted, then $\beta$ must be asserted". Here are employed Greek variables, for it cannot be said: "if *Cpq* is asserted, and *p* is asserted, then *q* must be asserted". It would be an error to assert *Cpq* or *q*, since neither of these expressions is a true proposition, and only true propositions can be asserted. But we can use Greek variables, as we can assume that $\alpha$ and $\beta$ satisfy the condition of verifying C$\alpha\beta$ and $\alpha$.

There are other properties of Greek variables which deserve our attention. Not only nothing can be substituted for the Greek variables, but also the rule of substitution for the Latin variables must be cautiously used in theses where occur both Latin and Greek variables. We always must keep in mind that possible interpretations of $\alpha$ and $\beta$ may be affected in such theses by a substitution for the Latin variables. This clearly appears in the following example: Axiom *CpC*$\alpha\beta$ is verified by the interpretation $\alpha$ = *Cpp*, $\beta$ = *p*, as *CpCCppp* is a thesis. Therefore

(16.14) *CpCCCsq*$\alpha$*CC*$\beta$*sCCsqs*

is also verified by this interpretation yielding the thesis:

(17.1) *CpCCCsqCppCCpsCCsqs*.

if, however, thesis (16.14) were altered by a substitution, for instance thus:

(16.14) *p*/*Cpp*, *s*/*p*, *q*/*p* * (III)

(III) *CCppCCCpp*$\alpha$*CC*$\beta$*pCCppp*,

then we would get from it by the above interpretation a false expression:

(IV) *CCppCCCppCppCCppCCppp*,

as all the antecedents of this expression are theses, and the consequent is a variable. The substitution: (16.14) *p*/*Cpp*, *s*/*p*, *q*/*p*, affects the possible interpretation $\alpha$ = *Cpp* and $\beta$ = *p*, because the variable *p*, for which the substitution is made, occurs in the interpretation too. Variables $\alpha$ and $\beta$ occurring in (III) are not the same as in (16.14). The same substitution applied to (17.1) gives:

(17.2) *CCppCCCppCCppCppCCCpppCCppp*,

which is different from (III) and a thesis. In our deduction of *CpCqp* no substitution has been made in theses with Greek variables. This is the safest proceeding. There are, however, theses with Latin and Greek variables, where a substitution for the Latin variables does not lead to an error. Of such a kind are theses which are verified for all interpretations of the Greek variables occuring in them, for instance substitutions of theses without Greek letters, like (16.13) or (16.15) of our deduction. Also thesis (16.18) belongs to this kind, as it does not contain $\alpha$, but only $\beta$. Now, for the interpretation $\alpha = Cp\beta$ the axiom $CpC\alpha\beta$ becomes a thesis, $CpCCp\beta\beta$, and therefore all the consequences of the set $CpCCp\beta\beta$, *CCpqCCqrCpr* and *CCCpqpp* are theses, whatever may be the meaning of $\beta$.

There are instances of expressions with Latin and Greek variables where a substitution for the Latin variables is unavoidable. Let us consider the following example:

I found a theorem concerning the principle of the syllogism which reads: "If $\alpha$ verifies the premisses $CCpq\alpha$ and $C\alpha CCqrCpr$, we can deduce from them *CCpqCCqrCpr*". In all interpretations of $\alpha$ which verify the premisses must occur both *p* and *q*, otherwise $CCpq\alpha$ were verified only when $\alpha$ is a thesis, but in this case *CCqrCpr* were a thesis too, and this is impossible. It is plain therefore that any substitution for *p* and *q* affects $\alpha$, and without such a substitution we cannot perform the proof. As we do not know how $\alpha$ is altered by a substitution, we must rely on the evident principle that two identical substitutions alter $\alpha$ in the same way. This principle is employed in the following deduction:

(17.3) $CCpq\alpha$

(17.4) $C\alpha CCqrCpr$

  (17.3) $p/Cpq,\ q/\alpha,\ r/CCqrCpr$ * C(17.3)–(17.5)

(17.5) $\alpha(p/Cpq,\ q/\alpha,\ r/CCqrCpr)$

  (17.4) $p/Cpq,\ q/\alpha,\ r/CCqrCpr$ * C(17.5)–C(17.4)–(11.1)

(11.1) *CCpqCCqrCpr.*

Although *r* does not occur in (17.3), the substitution for *r* has to be taken into account, since the same substitution must be made in (17.3) as in (17.4), and *r* may occur in an interpretation of $\alpha$. This substitution, of course, has no effect, when the interpretation of $\alpha$ does not contain *r*. The thesis arising by substitution from $\alpha$ is marked by $\alpha$ followed by the generating substitution in brackets. Two examples will clear up the matter:

First example: $\alpha = CCCpqrr$, i.e. *r* occurs in $\alpha$.

(17.6) *CCpqCCCpqrr*

(17.7) *CCCCpqrrCCqrCpr*

  (17.6) *p* / *Cpq*, *q* / *CCCpqrr*, *r* / *CCqrCpr* * C(17.6)–(17.8)

(17.8) *CCCCpqCCCpqrrCCqrCprCCqrCpr*

  (17.7) *p* / *Cpq*, *q* / *CCCpqrr*, *r* / *CCqrCpr* * C(17.8)–C(17.7)–(11.1)

(11.1) *CCpqCCqrCpr*

Second example: $\alpha = CCCqspq$, i.e. *r* does not occur in $\alpha$.

(17.9) *CCpqCCCqspq*

(17.10) *CCCCqspqCCqrCpr*

  (17.9) *p* / *Cpq*, *q* / *CCCqspq*, *r* / *CCqrCpr* * C(17.9)–(17.11)

(17.11) *CCCCCCqspqsCpqCCCqspq*

  (17.10) *p* / *Cpq*, *q* / *CCCqspq*, *r* / *CCqrCpr* *
    C(17.11)–C(17.10)–(11.1)

(11.1) *CCpqCCqrCpr*

The substitution for *r* in (17.9) is of no consequence, as *r* does not occur in (17.9). Expressions (17.6), (17.7), (17.9) and (17.10) are theses (try to prove them), the first two and the last belong to positive logic, and (17.9) is related to (15.3) which does not belong to positive logic.

We see from the deduction of section 16, and from the just proved theorem about the principle of the syllogism, that the introduction of Greek variables enables us to generalize connexions between theses of propositional logic.

## B. Negation

### 18. The principle of Duns Scotus

The most important function of propositional logic after implication is negation. We shall later see that by means of these two functions all the other functions of propositional logic can be defined. The negation functor which is a proposition–building functor of one propositional expression is denoted in my symbolic by "*N*". It is difficult to render the function "*Np*" either in English or in any other modern language, as there exists no single word for the propositional negation. The ancient Stoics used for this purpose the single word ουχι. We have to say by circumlocution "it–is–not–true–that" or "it–is–not–the–case–that". For the sake of brevity I shall use the expression "not", so that "*Np*" will mean "not–*p*".

There exists no thesis with *N* as the sole functor; another functor of two arguments is always necessary to build up a thesis with *N*. If this other functor is *C*, we have a "*C–N*–thesis", and these theses will be considered in the sequel.

To the most important *C–N*–theses belong the following two which are equivalent to each other by commutation, and represent the same idea:

(18.1) *CpCNpq*

(18.2) *CNpCpq*.

In words: "if *p*, then if not–*p*, then *q*", and "if not–*p*, then if *p*, then *q*". In both cases *q* depends upon two premisses, *p* and *Np*. Two propositions of the form $\alpha$ and $N\alpha$ are called contradictory, and a famous principle stated by Aristotle says that two contradictory propositions are never together true. Philosophers regard this principle of "excluded contradiction" as one of the most fundamental laws of thought. We shall see, however, that it has but a little importance in the system of propositional logic. The venom attributed usually to contradiction is hidden in the above theses (18.1) and (18.2). If two contradictory propositions, like $\alpha$ and $N\alpha$, were asserted both, we would get from each of these theses by two detachments an arbitrary proposition *q*, and therefore all possible propositions. A difference between truth and falsity would cease to exist, and it would be to no purpose to construct a system of logic.

Each of the above two theses may be called "the principle of Duns Scotus", as the idea involved in them was set forth in a commentary on Aristotelian logic ascribed to Duns Scotus. It is worth–while to reproduce his argument. It is based on some evident rules of inference concerning conjunction and alternation. Let us suppose, that the following conjunction consisting of two contradictory propositions: "Socrates is and Socrates is not" (*Socrates est et Socrates non est*) is true. If a conjunction is true, then each of its components is true. It follows therefore that the proposition "Socrates is" is true. From this proposition follows the alternation: "Socrates is or the stick stands in the corner" (*Socrates est vel baculus stat in angulo*) for an alternation is true, if one of its components is true. On the other side it follows from the conjunction that the proposition "Socrates is not", which is the negation of "Socrates is", is also true; but from an alternation and the negation of one of its components results the other component. Therefore it is true that "the stick stands in the corner". This conclusion which has no real connexion with the premisses shows that from two contradictory propositions may be formally deduced any proposition whatever.

The already quoted scholastic principle: *ad falsum sequitur quodlibet*, is based on the principle of Duns Scotus. Let us suppose that $\alpha$ is true; then we get from (18.1) by detachment $CN\alpha q$, where $N\alpha$ as the negation of a true proposition is false. If therefore the antecedent of an implication is false, the implication is true.

The two forms of Duns Scotus' principle are independent of each other. Thesis (18.2) is undetachable, as easily follows from its structure as well as from the undetaching matrix $M_9$, which verifies (18.2) The figures in the column $N$ are the values of $Np$ for arguments exhibited in the column $C$. $M_9$ does not verify (18.1). On the contrary, matrix $M_{10}$ verifies (18.1) without verifying (18.2), because we get for $p/2$, $q/3$: $CN2C23 = C12 = 2$.

| C | 1 2 | N |
|---|---|---|
| *1 | 2 2 | 2 |
| 2 | 1 1 | 2 |

$M_9$

| C | 1 | 2 | 3 | N |
|---|---|---|---|---|
| *1 | 1 | 2 | 2 | 3 |
| 2 | 1 | 1 | 2 | 1 |
| 3 | 1 | 1 | 1 | 1 |

$M_{10}$

Two important consequences result from the two forms of the principle of Duns Scotus and the first form of the principle of the syllogism:

(11.1) *CCpqCCqrCpr*

(18.1) *CpCNpq*

(11.1) $q/CNpq$ * C(18.1)–(18.3)

(18.3) *CCCNpqrCpr*

*

(18.2) *CNpCpq*

(11.1) $p/Np$, $q/Cpq$ * C(18.2)–(18.4)

(18.4) *CCCpqrCNpr.*

A curious example can be given as illustration of thesis (18.4). Let us assume that "if the positions and the velocities of all particles of a material system are determinated at a certain moment (= *p*), then they are determinated at the next moment too (= *q*). If this implication is true, then we may say: "the principle of determinism is valid (= *r*). As *r* follows from the implication *Cpq*, then *CCpqr* or the antecedent of (18.4), is in our example true; therefore its consequent, *CNpr*, must also be true. Let us now suppose according to the well-known Heisenberg's relation that "the positions and velocities of all particles of a material system are not determinated at a certain moment"; then *Np* is true, and therefore *r* is true, i.e. the conclusion "the principle of determinism is valid" remains true.

**19. The principle of Clavius**

Another very important *C–N*–thesis is the following principle:

(19.1) *CCNppp.*

In words: "If (if not–*p*, then *p*), then *p*". *CNpp* is the antecedent, *p* is the consequent. *Np* and *p* are two contradictory propositions, and cannot be together true. If therefore an implication of the form *CNpp* is true, *Np* cannot be true, for from *CNpp* and *Np* would follow by detachment *p*, and two contradictory propositions, *Np* and *p*, were together true.

I called *CCNppp* "the principle of Clavius", because Clavius first drew attention to this peculiar thesis. Clavius, a learned Jesuit of the XVIth century and one of the constructors of the Gregorian calendar, was a commentator of Euclid, and found that Euclid employed this thesis in one of his proofs. Euclid states this fundamental arithmetical theorem: "if the product of two integers, $a \times b$, is divisible by a prime number $n$, and $a$ is not divisible by $n$, then $b$ should be divisible by $n$". This theorem he applies then to a particular case, where $a$ equals $b$, and proves: "if the product $a \times a$ is divisible by a prime number $n$, then $a$ is divisible by $n$". The proof runs thus: "suppose that $a \times a$ is divisible by $n$; then if $a$ is not divisible by $n$, it is divisible by $n$; therefore it is divisible by $n$". An implication is here given which has as its antecedent the negation of its consequent; "if $a$ is not divisible by $n$, then $a$ is divisible by $n$", and a logical principle enables us to deduce from this implication its consequent. This logical principle was seen by Clavius, and this is his merit, because Aristotle himself did not see it, when he said that a consequence of the form "if *B* is not great, then *B* is great" is impossible.

It seems that the principle of Clavius has got some popularity among the learned Jesuits of the next two centuries, and was called by them *consequentia mirabilis*. A Jesuit mathematician, Girolamo Saccheri, tried in the beginnings of XVIIIth century to apply this principle to geometry. He thought that every fundamental principle must have the property to result from its own negation, and believed therefore that he could prove Euclid axiom of parallels by the *consequentia mirabilis*. He started then his investigations from the denial of this axiom thinking that this way would lead him to its proof. He did not succeed in proving the axiom but constructed without knowing it the first system of a non–Euclidean geometry,

The principle of Clavius has its consequence in a thesis mentioned already in the "Introduction":

(19.2) *CCpNpNp*.

This thesis is called in *Principia mathematica* the "principle of *reductio ad absurdum*". Although similar to the principle of Clavius, it is in some sense weaker. As we shall see later, the set of theses:

(11.1) *CCpqCCqrCpr*

(18.1) *CpCNpq*

(19.1) *CCNppp*

forms a sufficient basis for all the *C–N*–theses, and together with definitions, for the whole Theory of Deduction. Not so the set of theses which arises by replacing *CCNppp* by *CCpNpNp*. This latter set is verified by the matrix $M_{10}$, whereas *CCNppp* is not verified by this matrix.

**20. The two principles of double negation**

Two negations annul each other. This evident fact finds its expression in this following two theses called the "principles of double negation":

(20.1) *CpNNp*

(20.2) *CNNpp*.

In words: "If $p$, then not–not–$p$" and "if not–not–$p$, then $p$".

| C | 1 | 2 | N |
|---|---|---|---|
| *1 | 1 | 2 | 1 |
| 2 | 1 | 1 | 1 |

$M_{11}$

| C | 1 | 2 | N |
|---|---|---|---|
| *1 | 1 | 2 | 1 |
| 2 | 1 | 1 | 2 |

$M_{12}$

The two theses together form an equivalence. They are independent of each other, as can be proved by the matrices $M_{11}$ and $M_{12}$. $M_{11}$ verifies (20.1) without verifying (20.2), whereas $M_{12}$ verifies (20.2) without verifying (20.1). For the sake of convenience I shall say in the sequel that a matrix which does not verify a thesis "falsifies" it. $M_{11}$ verifies (20.1), but falsifies (20.2), $M_{12}$ verifies (20.2), but falsifies (20.1). The implicational part of both these matrices is the same; I denote it by $M_0$. $M_0$ is the simplest matrix that verifies the axiom of the implicational logic: (16.1) *CCCpqrCCrpCsp*, and therefore all the *C*–theses. If we define *N*1 by 2 and *N*2 by 1, and add this matrix for *N* to $M_0$, we get $M_{00}$, the simplest matrix that verifies all the *C*–*N*–theses.

| C | 1 | 2 |
|---|---|---|
| *1 | 1 | 2 |
| 2 | 1 | 1 |

$M_0$

| C | 1 | 2 | N |
|---|---|---|---|
| *1 | 1 | 2 | 2 |
| 2 | 1 | 1 | 1 |

$M_{00}$

The matrices $M_{11}$, $M_{12}$ and $M_{00}$ differ only by various combinations of the values 1 and 2 for *N*, viz. 11, 22, and 21. There remains only one combination not yet taken into consideration, 12. This combination yields the matrix $M_{13}$. $M_{13}$ verifies both principles of double negation, but falsifies the principle of Duns Scotus as well as the principle of Clavius.

| C | 1 | 2 | N |
|---|---|---|---|
| *1 | 1 | 2 | 1 |
| 2 | 1 | 1 | 2 |

$M_{13}$

Every *C*–*N*–thesis containing a genuine *N* must be falsified by at least one of the matrices $M_{11}$, $M_{12}$ or $M_{13}$. I say that a thesis contains a "genuine *N*", when it cannot be obtained by substitution from a thesis without *N*. So, for instance, thesis *CpCNpq* contains a genuine *N*, as it cannot be obtained by substitution from a thesis without *N*, whereas thesis *CpCNqp* does not contain a genuine *N* because it can be got from *CpCqp* by the substitution *q*/*Nq*. It can be proved that if a *C*–*N*–thesis is verified by each of the matrices $M_{11}$, $M_{12}$ and $M_{13}$, it does not contain a genuine *N*. We may argue from that,

that if a $C$–$N$–thesis contains a genuine $N$, it is not verified by each of these matrices. Expression $CCpqCNpNq$, which is verified by the matrices $M_{11}$, $M_{12}$ and $M_{13}$, cannot be adduced as objection against this theorem, because it is not a $C$–$N$–thesis. It is falsified by $M_{00}$, and as all $C$–$M$–theses must be verified by this matrix, that is not verified by it is not a $C$–$N$–thesis.

**21. The four principles of transposition**

Two arguments of the foregoing section I shall repeat here *in forma*: First argument: From the implication "if $T$ is a $C$–$N$–thesis, $T$ is verified by $M_{00}$" another implication follows: "if $T$ is not verified by $M_{00}$, $T$ is not a $C$–$N$–thesis". Second argument: Suppose that $T$ is a $C$–$N$–thesis; from the implication: "if $T$ is verified by each of the matrices $M_{11}$, $M_{12}$, and $M_{13}$, $T$ does not contain a genuine $N$" another implication follows: "if $T$ contains a genuine $N$, $T$ is not verified by each of the matrices $M_{11}$, $M_{12}$ and $M_{13}$". Both arguments are instances of two principles of transposition, viz.:

(21.1) $CCpqCNqNp$

(21.2) $CCpNqCqNp$.

There are still two other similar principles:

(21.3) $CCNpqCNqp$

(21.4) $CCNpNqCqp$.

All these principles have a similar construction: the two arguments of the antecedent, $p$ and $q$, are transposed in the consequent, so that $q$ is first, and $p$ second; moreover, the arguments of the antecedent differ from the arguments of the consequent by an $N$, so that $p$ (and similarly $q$) in the antecedent corresponds to $Np$ ($Nq$) in the consequent, and $Np$ ($Nq$) in the antecedent corresponds to $p$ ($q$) in the consequent. Besides the principle of the syllogism, the principles of transposition are the most important and most frequently used instruments of proof.

The first principle of transposition was known to the Stoics as *modus tollens*; it reads in the Stoic form:

If $\alpha$, then $\beta$ .
But not–$\beta$.
Therefore not–$\alpha$.

It is used to refute hypotheses by facts. Suppose that $\alpha$ is a hypothesis in form of a general law, and $\beta$ a singular proposition about a fact resulting from $\alpha$. Then the implication $C\alpha\beta$ is true, and consequently $CN\beta N\alpha$ also must be true. If we find, however, that the fact which should result from the hypothesis is not verified by experience, so that $\beta$ is false and $N\beta$ true, the hypothesis is refuted, because from $CN\beta N\alpha$ and $N\beta$ follows $N\alpha$.

The third principle of transposition is very frequently used in mathematics to prove a theorem indirectly. Suppose that we have to prove indirectly the theorem $\alpha$. We start from its negation $N\alpha$, and deduce from it a false consequence $\beta$. Then the implication $CN\alpha\beta$ is true, and consequently $CN\beta\alpha$ also must be true. But $\beta$ is false, therefore $N\beta$ true, and from $CN\beta\alpha$ and $N\beta$ follows $\alpha$.

All the principles of transposition are falsified by the matrix $M_{13}$, moreover, (21.2) is falsified by $M_{12}$, (21.3) by $M_{11}$, and (21.4) by both $M_{11}$ and $M_{12}$. The first principle with both negations in the consequent is in some sense the weakest, the last principle with both negations in the antecedent [of] the strongest thesis. Roughly speaking, the more matrices falsify a thesis, the stronger is the thesis. Axiom (16.1) of the implicational logic which is a very strong thesis is falsified by all the implicational matrices $M_1$ till $M_8$ being only verified by $M_0$.

The relations of the principles of transposition to other *C*– and *C*–*N*–theses are very numerous. I shall quote only two compound principles of transposition resulting from the combination of the first principle of transposition with the second form of the principle of the syllogism:

(21.5) *CCpCqrCpCNrNq*,

(21.6) *CCpCqrCqCNrNp*.

These theses can be employed to prove the syllogistic moods called *Baroco* and *Bocardo* on the basis of the mood *Barbara*. Let *p* mean: "every *a* is *b*", *q* — "every *b* is *c*", and *r* — "every *a* is *c*". Then *Np*, or the negation of "every *a* is *b*" means the same as "some *a* is not *b*" and *Nr* means the same as "some *a* is not *c*". We get by this interpretation from the antecedent of (21.5), *CpCqr*, the implicational form of the mood *Barbara*: "if every *b* is *c*, then if every *a* is *b*, every *a* is *c*", which is equivalent to its normal conjunctional form: "if every *b* is *c* and every *a* is *b*, then every *a* is *c*". In the consequent we get from *CpCNrNq* the implicational form of the mood *Baroco*: "if every *b* is *c*, then if some *a* is not *c*, some *a* is not *b*", which is equivalent to its normal conjunctional form: "if every *b* is *c* and some *a* is not *c*, then some *a* is not *b*".

### 22. Frege's system of axioms

Frege is the author of the theory of deduction in its modern axiomatic form. His system (F1) is based on *C* and *N* as the primitive terms, and consists of six axioms:

| | | |
|---|---|---|
| | (10.1) *CpCqp* | principle of simplification |
| | (13.1) *CCpCqrCCpqCpr* | principle of Frege |
| | (12.1) *CCpCqrCqCpr* | principle of commutation |
| (F1) | | |
| | (20.1) *CpNNp* | 1st principle of double negation |
| | (20.2) *CNNpp* | 2nd principle of double negation |
| | (21.1) *CCpqCNqNp* | 1st principle of transposition. |

It was shown already that this system is not independent, as the principle of commutation follows from the first two axioms. But if we remove this superfluous principle, the remaining axioms form an independent set. The independence of the negative axioms is proved by the matrices $M_{11}$, $M_{12}$ and $M_{13}$, since $M_{12}$ falsifies only (21.1) while verifying the remaining axioms, $M_{11}$ falsifies only (21.2), and $M_{13}$ falsifies only (21.1). Of the axioms without negation, (10.1) is verified by $M_3$ which falsifies (13.1), and (13.1) is verified by $M_8$ which falsifies (10.1). We must add, however, to $M_3$ and $M_8$ the values for *N*, getting thus $M_{3.1}$ and $M_{8.1}$. Of the five Frege's axioms that remain after (12.1) has been removed, $M_{8.1}$ falsifies only (10.1), and $M_{3.1}$ falsifies only (13.1).

| *C* | 1 | 2 | 3 | *N* |
|---|---|---|---|---|
| *1 | 1 | 2 | 3 | 3 |
| 2 | 1 | 1 | 3 | 2 |
| 3 | 1 | 1 | 1 | 1 |

$M_{3.1}$

| C | 1 | 2 | 3 | 4 | N |
|---|---|---|---|---|---|
| *1 | 1 | 2 | 4 | 4 | 4 |
| 2 | 1 | 1 | 4 | 4 | 3 |
| 3 | 2 | 2 | 1 | 2 | 2 |
| 4 | 1 | 2 | 1 | 1 | 1 |

$M_{8.1}$

Frege's system of axioms which yields all the *C–N*–theses and through definitions the whole theory of deduction can be reduced to only three axioms, since his three negative theses may be replaced by the fourth principle of transposition *CCNpNqCqp*. This fact may be a proof that this fourth principle is stronger than the first. I shall show below that system (F2):

(10.1) *CpCqp*
(F2) (13.1) *CCpCqrCCpqCpr*
(21.4) *CCNpNqCqp*

is deductively equivalent to (F1), i.e. that from (F1) follows (F2), and conversely from (F2) follows (F1).

(a) From (F1) to (F2).

Our premisses are: axioms of the system (F1) and theses:

(11.9) *CCqrCCpqCpr*
(11.4) *CCpCqrCCsqCpCsr*,

which follow from (10.1) and (13.1). We have to deduce (21.4).

(21.1) *p* / *Np*, *q* / *Nq* * (22.1)
(22.1) *CCNpNqCNNqNNp*
(20.1) *p* / *q* * (22.2)
(22.2) *CqNNq*
(11.4) *p* / *CNpNq*, *q* / *NNq*, *r* / *NNp*, *s* / *q* * C(22.1)–C(22.2)–(22.3)
(22.3) *CCNpNqCqNNp*
(11.9) *q* / *NNp*, *r* / *p*, *p* / *q* * C(20.2)–(22.4)
(22.4) *CCqNNpCqp*
(11.9) *q* / *CqNNp*, *r* / *Cqp*, *p* / *CNpNq* * C(22.4)–C(22.3)–(22.5)
(22.5) *CCNpNqCqp*. (21.4)

(b) From (F2) to (F1).

Our premisses are: axioms of the system (F2) and theses:

(11.1) *CCpqCCqrCpr*
(11.7) *CCqsCCpqCCsrCpr*
(11.9) *CCqrCCpqCpr*
(12.1) *CCpCqrCqCpr*
(13.7) *CCCqprCpr*

which follow from (10.1) and (13.1). We have to deduce (20.1), (20.2) and (21.1).

(13.7) *q* / *Np*, *p* / *Nq*, *r* / *Cqp* * C(21.4)–(22.6)
(22.6) *CNqCqp*
(22.6) *q* / *p*, *p* / *Nq* * (22.7)
(22.7) *CNpCpNq*
(13.1) *p* / *Np*, *q* / *p*, *r* / *Nq* * C(22.7)–(22.8)
(22.8) *CCNppCNpNq*
(11.1) *p* / *CNpp*, *q* / *CNpNq*, *r* / *Cqp* * C(22.8)–C(21.4)–(22.9)

(22.9) *CCNppCqp*

(12.1) *p / CNpp*, *r / p* * C(22.9)–(22.10)

(22.10) *CqCCNppp*

(22.10) *q / CqCCNppp* * C(22.10)–(22.11)

(22.11) *CCNppp*

(11.1) *p / Nq*, *q / Cqp* * C(22.6)–(22.12)

(22.12) *CCCqprCNqr*

(22.12) *q / Np*, *r / p* * C(22.11)–(22.13)

(22.13) *CNNpp* (20.2)

(22.13) *p / Np* * (22.14)

(22.14) *CNNNpNp*

(21.4) *p / NNp*, *q / p* * C(22.14)–(22.15)

(22.15) *CpNNp* (20.1)

(22.15) *p / q* * [(22.16)]

(22.16) *CqNNq*

(11.9) *r / NNq* * C(22.16)–(22.17)

(22.17) *CCpqCpNNq*

(11.1) *p / NNp*, *q / p*, *r / NNq* * C(22.13)–(22.18)

(22.18) *CCpNNqCNNpNNq*

(21.4) *p / Np*, *q / Nq* * (22.19)

(22.19) *CCNNpNNqCNqNp*

(11.7) *q / CpNNq*, *s / CNNpNNq*, *p / Cpq*, *r / CNqNp* *
C(22.18)–C(22.17)–C(22.19)–(22.20)

(22.20) *CCpqCNqNp*.

Theses (10.1) and (13.1) are the basis of positive logic. It suffices therefore to add to the positive logic the fourth principle of transposition in order to get the whole theory of deduction. As the implicational logic which is wider than the positive logic may be built up on the single axiom (16.1), the set of the following two axioms:

(16.1) *CCCpqrCCrpCsp*

(21.4) *CCNpNqCqp*

also is a sufficient basis of the theory of deduction.

### 23. On single axioms of the *C–N*–logic

In continuation of the above remarks about axioms of the *C–N*–logic I shall here mention the attempts that were made to establish this logic on only one axiom. The first single axiom of this kind was found by Tarski in 1925. It contained 53 letters and was inorganic. A thesis is called "inorganic", when some of its parts are theses. So, for instance, thesis *CrCpCqp* is inorganic, as its part *CpCqp* is a thesis. The first organic axiom was discovered by Sobociński in 1927, and contained 139 letters. Sobociński succeeded in 1933 to find an organic axiom of 27 letters, and his axiom was abbreviated by myself in 1936 to 25 letters. It reads *CCCpqCCCNrNstrCvCCrpCsp*, as is connected with the shortest axiom of the implicational logic. The deductions, however, from this axiom are very long and difficult. In 1937 I found another axiom of 23 letters:

(23.1) *CCCpqCCNrsCNttCCtpCvCrp*,

and from this axiom I shall now deduce the set of theses *CCpqCCqrCpr*, *CpCNpq* and *CCNppp*, as this set forms a sufficient basis of the *C–N*–logic.

(23.1) *t* / *Ns*, *v* / *NCrp* * (23.2)
(23.2) *CCCpqCCNrsCNNsNsCCNspCNCrpCrp*
(23.1) *p* / *Cpq*, *q* / *CCNrsCNNsNs*, *r* / *s*, *s* / *p*, *t* / *Crp* * C(23.2)–(23.3)
(23.3) *CCCrpCpqCvCsCpq*
(23.3) *v* / *Cpq* * (23.4)
(23.4) *CCCrpCpqCCpqCsCpq*
(23.3) *r* / *Crp*, *p* / *Cpq*, *q* / *CsCpq*, *v* / (23.1), *s* / (23.1) *
C(23.4)–C(23.1)–C(23.1)–(23.5)
(23.5) *CCpqCsCpq*
(23.5) *p* / *Nq*, *s* / *CNrs* * (23.6)
(23.6) *CCNqqCCNrsCNqq*
(23.1) *p* / *Nq*, *t* / *q* * C(23.6)–(23.7)
(23.7) *CCqNqCvCrNq*
(23.7) *v* / *CNrs*, *r* / *NNq* * (23.8)
(23.8) *CCqNqCCNrsCNNqNq*
(23.1) *p* / *q*, *q* / *Nq*, *t* / *Nq* * C(23.8)–(23.9)
(23.9) *CCNqqCvCrq*
(23.5) *p* / *CNqq*, *q* / *CvCrq* * C(23.9)–(23.10)
(23.10) *CsCCNqqCvCrq*
(23.10) *s* / *Cpq*, *v* / *NCrq* * (23.11)
(23.11) *CCpqCCNqqCNCrqCrq*
(23.1) *r* / *q*, *s* / *q*, *t* / *Crq* * C(23.11)–(23.12)
(23.12) *CCCrqpCvCqp*
(23.12) *r* / *Cpq*, *q* / *CCNrsCNtt*, *p* / *CCtpCvCrp*, *v* / (23.1) *
C(23.1)–C(23.1)–(23.13)
(23.13) *CCCNrsCNttCCtpCvCrp*
(23.12) *r* / *CNrs*, *q* / *CNtt*, *p* / *CCtpCvCrp*, *v* / (23.1) * C(23.13)–C(23.1)–(23.14)
(23.14) *CCNttCCtpCvCrp*
(23.12) *r* / *p*, *p* / *CsCpq*, *v* / (25.1) * C(23.5)–C(23.1)–(23.15)
(23.15) *CqCsCpq*
(23.15) *q* / *Crq* * (23.16)
(23.16) *CCrqCsCpCrq*
(23.12) *p* / *CsCpCrq*, *v* / (23.1) * C(23.16)–C(23.1)–(23.17)
(23.17) *CqCsCpCrq*
(23.17) *q* / *Cpq*, *s* / *CNss*, *p* / *NCrCpq* * (23.18)
(23.18) *CCpqCCNssCNcpCrqCpCrq*
(23.1) *r* / *s*, *t* / *CrCpq* * C(23.18)–(23.19)
(23.19) *CCCrCpqpCvCsp*
(23.19) *v* / *CNvs*, *s* / *Np* * (23.20)
(23.20) *CCCrCpqpCCNvsCNpp*
(23.1) *p* / *CrCpq*, *q* / *p*, *r* / *v*, *t* / *p* * C(23.20)–(23.21)
(23.21) *CCpCrCpqCvCvCrCpq*
(23.14) *v* / *CNtt* * (23.22)
(23.22) *CCNttCCtpCCNttCrp*
(23.21) *p* / *CNtt*, *r* / *Ctp*, *q* / *Crp*, *v* / (23.1) *
C(23.22)–C(23.1)–C(23.1)–(23.23)
(23.23) *CCtpCCNttCrp*

(23.23) $t/Ctp, p/CCNttCrp, r/s$ * C(23.23)–(23.24)

(23.24) $CCNCtpCtpCsCCNttCrp$

(23.24) $t/r, p/t, s/NCCNrrCNtt, r/Nt$ * (23.25)

(23.25) $CCNCrtCrtCNCCNrrCNttCCNrrCNtt$

(23.13) $s/r$ * (23.26)

(23.26) $CCCNrrCNttCCtpCvCrp$

(23.26) $r/Crt, t/CCNrrCNtt, p/CCtpCvCrp, v/$(23.1) *
C(23.25)–C(23.26)–C(23.1)–(23.27)

(23.27) $CCrtCCtpCvCrp$

(23.27) $v/Crt$ * (23.28)

(23.28) $CCrtCCtpCCrtCrp$

(23.21) $p/Crt, r/Ctp, q/Crp, v/$(23.1) *
C(23.28)–C(23.1)–C(23.1)–(23.29)

(23.29) $CCtpCCrtCrp$

(23.9) $q/p, v/r, r/CNpp$ * (23.30)

(23.30) $CCNppCrCCNppp$

(23.21) $p/CNpp, q/p, v/$(23.1) * C(23.30)–C(23.1)–C(23.1)–(23.31)

(23.31) $CrCCNppp$

(23.31) $r/CrCCNppp$ * C(23.31)–(23.32)

(23.32) $CCNppp$ (19.1)

(23.29) $t/CNpp$ * C(23.32)–(23.33)

(23.33) $CCrCNppCrp$

(23.29) $t/CrCNpp, p/Crp, r/q$ * C(23.33)–(23.34)

(23.34) $CCqCrCNppCqCrp$

(23.27) $r/p, t/q, p/r, v/NCpr$ * (23.35)

(23.35) $CCpqCCqrCNCprCpr$

(23.34) $q/Cpq, r/Cqr, p/Cpr$ * C(23.35)–(23.36)

(23.36) $CCpqCCqrCpr$ (11.1)

(23.5) $p/Np, s/NCNpq$ * (23.37)

(23.37) $CCNpqCNCNpqCNpq$

(23.33) $r/CNpq, p/CNpq$ * C(23.37)–(23.38)

(23.38) $CCNpqCNpq$

(23.13) $r/p, s/q, t/CNpq, p/CNpq, v/$(23.1) *
C(23.37)–C(23.38)–C(23.1)–(23.39)

(23.39) $CpCNpq$ (18.1)

Theses (23.32), (23.36) and (23.39) are a basis of the *C–N*–logic. It is not known, whether a shorter *C–N*–axiom exists, but this is probable. The final solution of this problem, i.e. the discovery of the shortest *C–N*–axiom, requires a great technical skill, and hardly could be made without some new conceptions of a general kind. The problem, however, has lost much of its importance, since it has been shown that by introduction of variable functors into the theory of deduction a system wider than this theory may be built up on a single axiom of only ten, and even of six letters.

### 24. The principles of the constructive and destructive dilemma

The Greek word "lemma" means, roughly speaking, a premiss, and "dilemma" means literally an argument which starts from two premisses. When both premisses concur to establish a conclusion, the dilemma is called "constructive", when they conor to refute a conclusion, the dilemma is called "destructive". In both arguments occur contradictory propositions: the constructive dilemma is connected with the principle of excluded middle, the destructive dilemma with the principle of excluded contradiction. The constructive dilemma reads in symbols:

(24.1) *CCpqCCNpqq*.

In words: "If (if *p*, then *q*), then [if (if not–*p*, then *q*), then *q*]". The two premisses are *Cpq* and *CNpq*; the conclusion is *q*. Let us suppose that the implications $C\alpha\beta$ and $CN\alpha\beta$ are both true. Now $\alpha$ and $N\alpha$ are contradictories, and according to the principle of excluded middle of two contradictory propositions one must be true. If $\alpha$ is true, then from $C\alpha\beta$ and $\alpha$ follows by detachment $\beta$; if $N\alpha$ is true, from $CN\alpha\beta$ and $N\alpha$ again follows $\beta$. Therefore $\beta$ is true.

The destructive dilemma reads:

(24.2) *CCpqCCpNqNp*

In words: "If (if *p*, then *q*), then if [(if *p*, then not–*q*), then not–*p*]". The two premisses are *Cpq* and *CpNq*; the conclusion is *Np*. Let us suppose that the implications $C\alpha$ and $C\alpha N\beta$ are both true. As $\beta$ and $N\beta$ are contradictories, one of them must be false according to the principle of excluded contradiction. Now, whether $\beta$ is false or $N\beta$, in any case $\alpha$ leads to a false consequence, and therefore must be false.

The principle of constructive dilemma occurs in Hilbert's set of axioms of the *C–N–* logic. The affirmative axioms of this set (which form a basis of positive logic and are independent of each other) were already mentioned in section 14. I repeat them below adding to them the two negative axioms given by Hilbert:

| | |
|---|---|
| (10.1) *CpCqp* | principle of simplification |
| (11.9) *CCqrCCpqCpr* | second form of the syllogism |
| (12.1) *CCpCqrCqCpr* | principle of commutation |
| (13.1) *CCpCpqCpq* | Hilbert's principle |
| (18.1) *CpCNpq* | principle of Duns Scotus |
| (24.1) *CCpqCCNpqq* | princ[iple] of constructive dilemma |

This set of axioms is, like Frege's set, not independent. From the negative axioms we can easily obtain by the principle of commutation the principle of Hilbert:

(12.1) *qNp*, *r*/*q* * C(18.1)–(24.3)

(24.3) *CNpCpq* (18.2)

(12.1) *p*/*Cpq*, *q*/*CNpq*, *r*/*q* * C(24.1)–(24.4)

(24.4) *CCNpqCCpqq*

(24.4) *q*/*Cpq* * C(23.3)–(13.11)

(13.11) *CCpCpqCpq*.

If we remove the redundant axiom (13.11), all the remaining five axioms are probably independent of each other. I say "probably", because I was not able either to deduce the principle of commutation from the other axioms, or to show that it is independent of them. The independence of the principle of simplification of the remaining axioms including the redundant principle of Hilbert is proved by the matrix $M_{1.1}$ which falsifies both identity and simplification, while verifying all the other principles. The independence of the syllogism of all the other axioms including Hilbert's principle is proved by the matrix $M_{4.1}$; $M_{11}$ falsifies only axiom (18.1), and $M_{12}$ only the last axiom. If we replace the second form of the syllogism by its first form, as was done by von Neumann, then we can derive the principle of commutation.

| C | 1 | 2 | 3 | N |
|---|---|---|---|---|
| *1 | 1 | 2 | 2 | 2 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 2 | 2 | 2 |

$M_{1.1}$

| C | 1 | 2 | 3 | N |
|---|---|---|---|---|
| *1 | 1 | 1 | 3 | 3 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 |

$M_{4.1}$

The same result we may obtain by replacing the principle of constructive dilemma by its other form:

(24.4) *CCNpqCCpqq*,

which yields in connexion with (18.1) by means of the syllogism the principle of assertion *CpCCpqq*, and consequently the principle of commutation.

With the principle of constructive dilemma are connected two important principles not having a special name:

(24.5) *CCprCCqrCCNpqr*

(24.6) *CCNprCCqrCCpqr*.

They may be called "principles of constructive trilemma", as from each of them follows the conclusion *r* by three premisses, from (24.5) by *Cpr*, *Cqr* and *CNpq*, and from (24.6) by *CNpr*, *Cqr* and *Cpq*. By means of commutation and identity we can derive from them the two forms of the principle of constructive dilemma, thus:

The premises:

(9.1) *Cpp*

(12.1) *CCpCqrCqCpr*

From (24.5):

(24.5) *r*/*q* * (24.7)

(24.7) *CCpqCCqqCCNpqq*

(9.1) *p*/*q* * (24.8)

(24.8) *Cqq*

(12.1) *p*/*Cpq*, *q*/*Cqq*, *r*/*CCNpqq* * C(24.7)–C(24.8)–(24.1)

(24.1) *CCpqCCNpqq*

From (24.6):

(24.6) *r*/*q* * (24.9)

(24.9) *CCNpqCCqqCCpqq*

(12.1) *p*/*CNpq*, *q*/*Cqq*, *r*/*CCpqr* * C(24.9)–C(24.8)–(24.4)

(24.4) *CCNpqCCpqq*.

Principle (24.6) forms together with thesis (24.10) which follows from (13.14) by a change of variables:

(13.14) *q*/*p*, *p*/*q* * (24.10)

(24.10) *CCCpqrCqr*,

and thesis (18.4) a set of three axioms:

(18.4) *CCCpqrCNpr*

(24.10) *CCCpqrCqr*

(24.6) *CCNprCCqrCCpqr*

which is a sufficient basis of the *C–N*–logic. This set of axioms is in some respect the most elegant of all, because it forms an organic whole. Only three different expressions are contained in it: *CCpqr* which is the antecedent of the first two axioms and the consequent of the third axiom, *CNpr* which is the consequent of the first and the antecedent of the third axiom, and *Cqr* which is the consequent of the second and the (second) antecedent of the third axiom. The three axioms imply together that the expression $CC\alpha\beta\gamma$ is deductively equivalent to the expressions $CN\alpha\gamma$ and $C\beta\gamma$.

The other "trilemmatic" thesis, (24.5), yields by means of a definition an important principle of alternation.

The survey of most important theses of the *C–N*–logic is herewith brought to the end.

## C. Alternation

### 25. The definition of alternation

We shall learn in the systematic part that in the two–valued theory of deduction there exist four functions of one argument, and sixteen functions of two arguments. All these functions can be defined by implication and negation, so that a system based on these terms may be called "semantically complete".

Not all of these functions are of equal importance. We shall consider in the sequel only those functions that have a corresponding term in the ordinary language, and may convey for this reason some new intuitive information. There are mainly four such functions, all of two arguments: alternation, conjunction, equivalence and disjunction. As all these functions have to be defined by our primitive terms, implication and negation, some preliminary remarks about definitions are necessary.

I agree with *Principia mathematica* that "a definition is a declaration that a certain newly–introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known". Every definition consists therefore of two parts: the *definiendum* and the *definiens*. The *definiendum* contains the newly–introduced symbol, the *definiens* is a combination of symbols already known. The authors of *Principia* express a definition by putting the *definiendum* to the left and the *definiens* to the right, with the sign "=" between, and the letters "Df" to the right of the definiens. The scheme of definitions accepted in *Principia* looks therefore thus:

(a) $A = B$ Df,

and has the sense: "*A* means by definition the same as *B*". As *A* means the same as *B*, we may everywhere replace *A* by *B*, and conversely *B* by *A*, so that, as the authors of *Principia* say, "theoretically, it is unnecessary ever to give a definition: we might always use the *definiens* instead".

The above method of writing down definitions has one disadvantage: it introduces into the system a new constant term, that of "definitional equality". It seems that the authors of *Principia* have felt this difficulty, as they would not include definitions into their system, saying "that a definition is, strictly speaking, no part of the subject it which it occurs". "Moreover", they continue, a definition "is not true or false, being the expression of a volition, not of a proposition". I cannot agree with the last two statements. Although the sentence "*A* means by definition the same as *B*" may be the result of a volition, nevertheless it is a proposition concerning the terms *A* and *B*, and like every proposition must be either true or false. On the other hand, as the authors of *Principia* admit themselves, "the definitions are what is most important, and what most deserves the reader's prolonged attention", and consequently cannot be excluded from the system. If we want therefore not to encumber the system with a new primitive term, we must look for another method of expressing definitions.

There exists such other method: we may employ equivalence to denote the relation between the definiens and the definiendum. Equivalence is a function of the theory of deduction, whereas the sign "= Df" accepted in *Principia* does not denote a function of our theory. But equivalence is not a primitive term of the *C*–*N*–logic, it must be first defined. So would arise a vicious circle, and consequently this way is also impassable.

I found a third method which avoids the above difficulties. I do not introduce a new constant term in order to express definitions, but a new kind of variable which I shall later use to extend the theory of deduction. It is a variable functor of one propositional expression denoted by δ. δ*p* represents any propositional expression containing *p*, for instance *Cpq*, *Cpp*, *p*, and so on. δ is a substitution–variable, and in order to obtain from δ*p* the expression *Cpq* I write δ/*C*'*q*, where the apostrophy "'" denotes an empty place that has to be filled up by the argument of δ. In a similar way I get from δ*p* the expression *Cpp* by the substitution δ/*C*'', and *p* by the substitution δ/'. A definition expressed by means of δ has the form:

(b) Cδ*A*δ*B*.

From this form we get without a new thesis, only by substitution and detachment the converse implication:

(c) Cδ*B*δ*A*.

The proof of this assertion is instructive, as it shows how works the rule of substitution for δ:

(b) δ/Cδ'δ*A* * (d)

(d) CCδ*A*δ*A*Cδ*B*δ*A*

(b) δ/CCδ*A*δ'Cδ*B*δ*A* * C(d)–C(b)–(c)

(c) Cδ*B*δ*A*.

Cδ*A*δ*B* means that *A* — wherever it occurs — may be replaced by *B*, and likewise Cδ*B*δ*A* means that *B* — wherever it occurs — may be replaced by *A*. It does not matter whether *A* is the *definiens* and *B* the *definiendum*, or conversely. I shall write the *definiens* in the first place, and the *definiendum* in the second. Plenty of examples given in the next sections will explain the matter thoroughly.

**26. The definition of alternation**

I mean by an "alternation" a combination of two propositions *p* and *q* of the form: "either *p* or *q*", where the words "either–or" ("either" may be omitted) have to be interpreted in the non–exclusive sense. That means: an alternation is true, when at least one of its components, *p* or *q*, is true (they may be both true), and it is false, only when both its components are false. I quote for explanation an example given by Keynes: "He has either used bad text–books or he has been badly taught". It is plain that the components of this alternation do not exclude each other.

There are two possible definitions of the alternation on the basis of the terms C and

N. In the *definiens* of one of them, there occur both functors C and *N*, in the other only C. I begin with the first definition which is more intuitive, than the second. It runs:

(26.1) *CδCNpqδApq*.

The newly–introduced symbol is *A* (alternation). *Apq* means in words: "(either) *p* or *q*". The definition states that *CNpq* may everywhere be replaced by *Apq*, and consequently *Apq* may everywhere be replaced by *CNpq*, because from (26.1) follows the converse implication according to the scheme of the foregoing section, thus:

(26.1) δ / *Cδ'δCNpq* * (26.2)

(26.2) *CδCδCNpqδCNpqCδApqδCNpq*

(26.1) δ / *CCδCNpqδ'CδApqδCNpq* * C(26.2)–C(26.1)–(26.3)

(26.3) *CδApqδCNpq*.

Since *CNpq* may everywhere be replaced by *Apq*, and conversely *Apq* by *CNpq*, *Apq* means the same as *CNpq*, or in words: "*p* or *q*" means the same as "if not–*p*, then *q*".

From the definition (26.1) and its complement (26.3) we get by a simple substitution, i.e. by omission of δ, the theses:

(26.1) δ/' * (26.4)

(26.4) *CCNpqApq*

(26.3) δ/' * (26.5)

(26.5) *CApqCNpq*.

Thesis (26.5) lies at the bottom of Duns Scotus' argument quotet in section 18. He correctly derives from the premisses: "either Socrates is or the stick stands in the corner (*Apq*)", and "Socrates is not (*Np*)", the conclusion (*q*): "the stick stands in the corner".

The second definition runs:

(26.6) *CδCCpqqδApq*.

According to this definition *Apq* means the same as *CCpqq*. This definition is not so clear, as the first. It may be explained as follows: The expressions *CCpqq* and *Apq* have the same meaning as they always are either both true, or both false. If *q* is true, then *Apq* is true, since an alternation is true when one of its components is true, and *CCpqq* is true according to the scholastic principle: *verum sequitur ad quodlibet*. If *q* is false and *p* is true, then *Apq* is true, but *Cpq* is false, and therefore *CCpqq* is true according to another scholastic principle: *ad falsum sequitur quodlibet*. If *q* is false and *p* is false, then *Apq* is false, since an alternation is false when both its components are false, and *CCpqq* is false, because *Cpq* is true according to the second scholastic principle and *q* is false. From the definition (26.6) and its complement:

(26.7) *CδApqδCCpqq*

we get the following two theses:

(26.6) δ/' * (26.8)

(26.8) *CCCpqqApq*

(26.7) δ/' * (26.9)

(26.9) *CApqCCpqq*.

These theses can be derived form (26.4) and (26.5) by means of some principles already known to us:

(11.1) *CCpqCCqrCpr*

(18.4) *CCCpqrCNpr*

(18.4) *r* / *q* * (26.10)

(26.10) *CCCpqqCNpq*

(11.1) *p* / *CCpqq*, *q* / *CNpq*, *r* / *Apq* * C(26.10)–C(26.4)–(26.8)

(26.8) *CCCpqqApq*

*

(24.4) *CCNpqCCpqq*

(11.1) *p* / *Apq*, *q* / *CNpq*, *r* / *CCpqq* * C(26.5)–C(24.4)–(26.9)

(26.9) *CApqCCpqq*.

In a subsequent part of this work I shall introduce the variable functor δ not only into definitions, but into the whole theory of deduction. Then I shall prove the following most important thesis:

(26.11) *CCpqCCqpCδpδq*.

For the moment the reader must take this thesis for granted. By means of (26.11) we may deduce the definition (26.6) from (26.3) and (26.9):

(26.11) *p* / *CCpqq*, *q* / *Apq* * C(26.8)–C(26.9)–(26.6)

(26.6) *CδCCpqqδApq*.

In the same way we may deduce by means of (26.11) the definition (26.1) from (26.4) and (26.5):

(26.11) *p* / *CNpq*, *q* / *Apq* * C(26.4)–C(26.5)–(26.1)

(26.1) *CδCNpqδApq*.

We see by these examples that our definitions expressed by means of δ are equivalent to two implications without δ, converse to each other.

### 27. The most important *C*–*A*–theses

All the theses of this section are simple translations of some *C*–*N*–theses obtained by help of the definition (26.1). We get immediately:

(19.1) *CCNppp*

(26.1) δ / *C'p*, *q* / *p* * (27.1)

(27.1) *CAppp*.

Thesis (27.1) is called in *Principia mathematica* the "principle of tautology". "Tautology" means "saying the same thing twice". Saying "*p* or *p*" we say *p* twice; it means the same, as to say "*p*", for "*p* or *p*" is equivalent to "*p*". The converse implication to (27.1), *CpApp*, may be proved as follows:

(18.1) *CpCNpq*

(26.1) δ / *Cp'* * (27.2)

(27.2) *CpApq*

(27.2) *q* / *p* * (27.3)

(27.3) *CpApp*.

*CpApp* gives together with *CAppp* the conclusion *CδAppδp*, and this conclusion states that "*p* or *p*" means the same as "*p*". The name of tautology given to thesis (27.2) seems to be justified.

Some authors, however, employ the word "tautology" in a sense that is not only not justified, but even misleading. They call all the theses of the theory of deduction "tautologies" connecting with this term certain bad philosophical speculations. They say, for instance, that the proposition "it is raining" conveys an information about an actual state of things, and so does the proposition "it is not raining". But the proposition "either it is raining or it is not raining" which is a substitution of the logical principle of excluded middle, does not convey any information at all, it is not meaningless indeed but empty, just a "tautulogy" like all logical laws. The principle of excluded middle which has the form *ApNp* is a thesis of our system, but I cannot understand why it is called a tautology. It does not say the same thing twice. The mistake to call it a tautology arose perhaps by a psychological association; who thinks that logical laws do not convey any real information, may them call tautologies, as saying the same thing twice does not convey a new information too. But it is wrong to expect that logical laws should give informations about an actual state of things, they are instruments of proof, not of experimental research. Moreover they are by no means empty, as they precise the meaning of some very important logical functors. It seems to me that all this talk about tautologies is but an attempt to reintroduce into logic under another name the old Kantian distinction between "analytic" and "synthetic judgements", and to show that logical laws are all analytic. This Kantian distinction, however, is of no importance for our system of logic, as it can be applied only to propositions with a subject and predicate. Theses of the theory of deduction have neither a subject nor a predicate, and the question whether they are analytic or synthetic is therefore futile.

Let us now derive some other *C*–*A*–theses:

(10.1) *CpCqp*

(10.1) *p* / *q*, *q* / *Np* * (27.4)

(27.4) *CqCNpq*

(26.1) δ / *Cq'* * C(27.4)–(27.5)

(27.5) *CqApq*.

Thesis (27.5) is called in *Principia* the "principle of addition", because it states, as the authors say, "that if a proposition is true, any alternative may be added without making it false". For the same reason thesis (27.2) nay be called by the same name. Duns Scotus employs this principle in his argument, when he says that from the proposition "Socrates is" follows the alternation "either Socrates is or the stick stands in the corner".

(21.3) *CCNpqCNqp*

(26.1) δ / *C'CNqp* * C(21.3)–(27.6)

(27.6) *CApqCNqp*

(26.1) δ / *CApq'*, *p* / *q*, *q* / *p* * C(27.6)–(27.7)

(27.7) *CApqAqp*.

Thesis (27.7) is called in *Principia* the "principle of permutation", because it states, as the authors say, the "permutative law for logical addition". Alternation is sometimes called "logical addition" in analogy to algebraical addition. This dates from the time when algebraical tendencies were prevailing in symbolic logic. It must be stressed that such loose analogies are to be rejected, as logic is not mathematics, and has its own problems and methods.

(12.1) *CCpCqrCqCpr*

(12.1) *p* / *Np*, *q* / *Nq* * (27.8)

(27.8) *CCNpCNqrCNqCNpr*

(26.1) δ / *C'CNqrCNpr*, *q* / *CNqr* * C(27.8)–(27.9)

(27.9) *CApCNqrCNqCNpr*

(26.1) δ / *CAp'CNqCNpr*, *p* / *q*, *q* / r * C(27.9)–(27.10)

(27.10) *CApAqrCNqCNpr*

(26.1) δ / *CApAqr'*, *p* / *q*, *q* / *CNpr* * C(27.10)–(27.11)

(27.11) *CApAqrAqCNp*

(26.1) δ / *CApAqrAq'*, *q* / *r* * C(27.11)–(27.12)

(27.12) *CApAqrAqApr*

Thesis (27.12) is called in *Principia* the "associative principle", again in analogy to the algebraical addition. It is a "primitive proposition" in *Principia*, i.e. an axiom, like the previous theses (27.1), (27.5), (27.7) and the next thesis (27.16). The authors explain that the proposition:

(27.13) *CApAqrAApqr*

which would be the natural form for the associative law, has less deductive power, and is therefore not taken as a primitive proposition. The importance of (27.12) is of a more historical than systematical nature.

(11.9) *CCqrCCpqCpr*

(11.9) *p* / *Np* * (27.14)

(27.14) *CCqrCCNpqCNpr*

(26.1) δ / *CCqrC′CNpr* * C(27.14)–(27.15)

(27.15) *CCqrCApqCNpr*

(26.1) δ / *CCqrCApq′*, *q* / *r* * C(27.15)–(27.16)

(27.16) *CCqrCApqApr*

Thesis (27.16) is called in *Principia* the "principle of summation", and is explained as follows: "In an implication, an alternative may be added to both premiss and conclusion without impairing the truth of the implication".

The last three *C*–*A*–theses, also very important, are not to be found in Principia:

(18.3) *CCCNpqrCpr*

(26.1) δ / *CC′rCpr* * C(18.3)–(27.17)

(27.17) *CCApqrCpr*

(13.8) *CCCqprCqr*

(13.8) *q* / *Np*, *p* / *q* * (27.18)

(27.18) *CCCNpqrCqr*

(26.1) δ / *CC′rCqr* * C(27.18)–(27.19)

(27.19) *CCApqrCqr*

(24.5) *CCprCCqrCCNpqr*

(26.1) δ / *CCprCCqrC′r* * C(27.19)–(27.20)

(27.20) *CCprCCqrCApqr*

Theses (27.17), (27.18) and (27.20) form an organic whole, like theses (18.4), (24.10) and (24.6) quoted in section 24. In a system of theory of deduction based on *A* and *N* as primitive terms they may replace the five axioms given in *Principia mathematica*.

**28. The system of axioms in** ***Principia mathematica***

The system of theory of deduction set forth in *Principia* is based on *A* and *N* as two primitive terms. It is semantically complete, for the *C*–*N*–system is semantically complete, and C can be defined by *A* and *N*. This definition is put on the head of the system, as the axioms are expressed not in primitive terms *A* and *N*, but in *A* and *C*. The system consists therefore of one definition and five axioms. The definition reads in symbols (the symbolism is mine):

(28.1) CδA*Npq*δ*Cpq*.

In words: ""if *p*, then *q*" means the same as "either not–*p* or *q*"". The five axioms are:

| | |
|---|---|
| (27.1) *CAppp* | the principle of tautology |
| (27.5) *CqApq* | the principle of addition |
| (27.7) *CApqAqp* | the principle of permutation |
| (27.11) *CApAqrAqApr* | the associative principle |

(27.16) *CCqrCApqApr* the principle of summation

It is, of course, not an error to employ a defined term in the axioms of a system, but it is more elegant to formulate the axioms in primitive terms, and not to begin with a definition. Moreover, implication is the most natural primitive term of the theory of deduction, because it is connected with the rule of detachment, and should therefore not be replaced by alternation. From this point of view the systems of Frege or Hilbert, being based on implication and negation, must be preferred. Besides, the *C–N*–systems set forth by myself consist of fewer axioms, and are consequently simpler. In my opinion, the system of theory of deduction, as exposed in *Principia*, is noteworthy today only from the historical standpoint.

The system of *Principia* is not independent. It is a strange coincidence that each of the historically important systems of the theory of deduction, that of Frege, of Hilbert and of Russell, contains one superfluous axiom. The reason is that proofs of independence in logic were not known to the authors of those systems, and, as we read in *Principia*, the recognized methods of proving independence were not applicable, without reserve, to fundamentals. A method suitable for this purpose was published in 1926 by Bernays who proved the associative principle is derivable from the other principles of *Principia*, whereas the remaining axioms are independent of each other. The same results and by the same methods were obtained by myself independently of Bernays, and were published before Bernays in 1925, but without proof.

I shall now show how axiom (27.11), i.e. the associative principle can be deduced from the definition (28.1) and the other axioms given in *Principia*.

(27.16) *p* / *Np* * (28.2)

(28.2) *CCqrCANpqANpr*

(28.1) δ / *CCqrC′ANpr* * C(28.2)–(28.3)

(28.3) *CCqrCCpqANpr*

(28.1) δ / *CCqrCCpq′*, *q* / *r* * C(28.3)–(28.4)

(28.4) *CCqrCCpqCpr*

(28.4) *q* / *Apq*, *r* / *Aqp*, *p* / *q* * C(27.7)–C(27.5)–(28.5)

(28.5) *CqAqp*

(28.5) *q* / *p*, *p* / *r* * (28.6)

(28.6) *CpCpr*

(27.5) *q* / *Apr*, *p* / *q* * (28.7)

(28.7) *CAprAqApr*

(28.4) *q* / *Apr*, *r* / *AqApr* * C(28.7)–C(28.6)–(28.8)

(28.8) *CpAqApr*

(27.5) *q* / *r* * (28.9)

(28.9) *CrApr*

(27.16) *q* / *r*, *r* / *Apr*, *p* / *q* * C(28.9)–(28.10)

(28.10) *CAqrAqApr*

(27.16) *q* / *Aqr*, *r* / *AqApr* * C(28.10)–(28.11)

(28.11) *CApAqrApAqApr*

(27.16) *q* / *p*, *r* / *AqApr*, *p* / *AqApr* * C(28.8)–(28.12)

(28.12) *CAAqAprpAAqAprAqApr*

(28.4) *q* / *App*, *r* / *p*, *p* / *q* * C(27.1)–(28.13)

(28.13) *CCqAppCqp*

(28.13) *q* / *AAqAprp*, *p* / *AqApr* * C(28.12)–(28.14)

(28.14) *CAAqAprpAqApr*

(27.7) *q* / *AqApr* * (28.15)

(28.15) *CApAqAprAAqAprp*

(28.4) *q* / *AAqAprp*, *r* / *AqApr*, *p* / *ApAqApr* *
C(28.14)–C(28.15)–(28.16)

(28.16) *CApAqAprAqApr*

(28.4) *q* / *ApAqApr*, *r* / *AqApr*, *p* / *ApAqr* * C(28.16)–C(28.11)–(28.17)

(28.17) *CApAqrAqApr*

The system of axioms of *Principia* can be replaced by the following set of axioms:

(27.17) *CCApqrCpr*

(27.19) *CCApqrCqr*

(27.20) *CCprCCqrCApqr*,

which forms an organic whole. We easily get from this set by the substitution *p* / *Np* and by help of the definition:

(28.1) *CδANpqδCpq*

the theses:

(18.4) *CCCpqrCNpr*

(24.10) *CCCpqrCqr*

(24.6) *CCNprCCqrCCpqr*,

which can be taken as basis of the *C*–*N*–logic.

**29. The principle of excluded middle**

This principle deserves a special consideration. It is not a *C*–*N*–, but a *A*–*N*–thesis, and can be deduced from the principle of identity:

(9.1) *Cpp*

(9.1) *p* / *Np* * (29.1)

(29.1) *CNpNp*

(27.1) *CδCNpqδApq* definition of *Apq*

(27.1) δ/', *q* / *Np* * C(29.1)–(29.2)

(29.2.) *ApNp* principle of excluded middle.

In words "either *p* or not *p*", According to an old philosophic doctrine which persists till today, the principle of excluded middle is held to belong with the principle of identity and the principle of excluded contradiction to the triplet of the so–called "fundamental laws of thought". Philosophers are seldom trained in exact thinking, and their opinions are seldom right. The principle of excluded middle is not a law of thought as no logical principle is a law of thought, and it is not a fundamental law, as it can be derived from other principles, which have therefore a better claim to be regarded as fundamental.

There exists, however, a principle which being really fundamental is often mixed up with the principle of excluded middle. It is a pity that not only philosophers, but also some logicians are sometimes careless in their expressions. So we may read, for instance, in *Principia* the following translation in words of the definition of implication: ""*p* implies *q*" is to be defined to mean "either *p* is false or *q* is true"". The great Polish logician, S. Leśniewski, one of the most exact thinkers I ever met, justly criticises such expressions. The symbolic expression *ANpq* (you can take the symbols of *Principia* instead) means only "either not–*p* or *q*", but never means "either *p* is false or *q* is true". The so–called "truth–values", i.e. truth and falsity, are not introduced as logical terms into the system of *Principia*. "$q$" and "$q$ is true" are different expressions, and so are "not–$p$" and "$p$ is false". Putting $p$ for $q$ in the above alternation we get an apparently true statement "either $p$ is false or $p$ is true", because it has to mean the same, as the true proposition "$p$ implies $p$". If this statement is true, it must be true for all $p$, so that there may be said: "Every proposition is either true or false". This last sentence embodies the so–called "principle of bivalence", which states that there are only two truth values, truth and falsity. We shall later see that the principle of bivalence is the deepest fundamental of our whole logic, and should not be mixed up with the principle of excluded middle.

Aristotle who first formulated the principle of excluded middle seems to accept in a very interesting chapter of his *Perihermen*[*e*]*ias* that in propositions referring to future contingent facts neither part of a contradiction is true or false. Suppose that a naval battle is such a future contingent fact. Aristotle says that the proposition "a naval battle will happen tomorrow" is today neither true nor false. For if it is true today, the naval battle must happen tomorrow, and if it is false today, the naval battle cannot happen tomorrow. In both cases the fact referred to ceases to be contingent, as contingent facts may but must not happen. Aristotle therefore seems to deny the principle of bivalence for propositions referring to future contingent facts. But he does not deny the principle of excluded middle for these facts, because he says explicitly that the proposition "a naval battle will happen tomorrow or it will not happen" is necessarily true.

Aristotle's indeterministic opinion about future contingent facts found followers among the Epicureans. There exists in Cicero's fragments *De fato* a passage on this subject very important for the history of logic. Of two contradictory propositions, Cicero says, one must be true, against Epicurus, and the other false; so it was true before all centuries that "Philoctetes will be healed", and it was false that "he will not be healed". Unless we liked to follow the opinion of the Epicureans who say that such propositions are neither true nor false; or, since to say this is a shame, they say, what is more shamless, that the disjunction formed of such propositions is true, whereas neither of its components is true. Cicero defeats this last opinion as unreasonable, and adheres to the principle of bivalence stated explicitly by Chrysippus that every proposition must be either true or false.

For a student of modern symbolic logic there is highly interesting to find in Cicero's works an opinion which is now shared by a large school of the so–called "intuitionist logic". Cicero regards as sham[e]less to contend that a disjunction, or rather an alternation, of contradictories, i.e. a proposition of the form "$p$ or not–$p$" may be true without having a true component. The same is maintained by Brouwer and Heyting, and their followers, the intuitionists. I shall later expound the intuitionist logic in a special section. For the moment it suffices to know, that the intuitionists accept as $C$–axioms the axioms of the positive logic, as $C$–$N$–axioms the theses $CpCNpq$ and $CCpqCCpNqNp$ (or $CCpNpNp$), as $A$–axioms the theses $CpApq$, $CqApq$ and $CCprCCqrCApqr$. $A$ is a primitive term like C and $N$, for the definitions of $A$ by means of C and $N$, and of C, are not recognized by them. The principle of excluded middle $ApNp$ is verified only when either $p$ is a true and asserted proposition, or $Np$. Suppose that $\alpha$ is asserted: we get $A\alpha N\alpha$ from $CpApq$ by the substitution $p/\alpha$, $q/N\alpha$ and by detachment; when N$\alpha$ is asserted, we get $A\alpha N\alpha$ from $CqApq$ by the substitution $q/N\alpha$, $q/\alpha$, and by detachment. $ApNp$ is not generally true, because neither $p$ can be asserted being a variable, nor $Np$. This reminds us of Cicero who feels that a proposition of the form *ApNp* should be rejected, when neither of its components is true.

Heyting who has formalized the intuitionist logic according to the ideas of Brouwer, gives a matrix in order to show that *ApNp* is independent of the axioms of the intuitionist logic. The axioms of the positive logic are verified by this matrix, the theses *CpCNpq* and *CCpqCCpNqNp* are verified too, and so are the *A*–axioms, *CpApq*, *CqApq*, and *CCprCCqrCApqr*. *ApNp* is falsified, for we have for $p/2$: *A*2*N*2 = *A*23 = 2. Also falsified are the theses: *CCCpqpp* (for $p/2$, $q/3$), *CCNppp* (for $p/2$), *CCNpqCNqp* (for $p/2$, $q/3$), *CCNpNqCqp* (for $p/2$, $q/1$), *CNNpp* (for $p/2$), and *CCpqCCNpqq* (for $p/2$, $q/2$). All these theses are rejected by the intuitionists. The definition (26.1) *C*δ*CNpq*δ*Apq* is falsified too, for we get from it by the substitution δ/′ the thesis *CCNpqApq*, and this thesis gives for $p/2$, $q/3$: *CCN*23*A*23 = *CC*322 = *C*12 = 2. In a similar way we get from the definition (26.6) *C*δ*CCpqq*δ*Apq* the thesis *CCCpqqApq*, which is falsified for $p/2$, $q/3$.

| *C* | 1 | 2 | 3 | *N* |
|---|---|---|---|---|
| *1 | 1 | 2 | 3 | 3 |
| 2 | 1 | 1 | 3 | 3 |
| 3 | 1 | 1 | 1 | 1 |

| *A* | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 |
| 3 | 1 | 2 | 3 |

$M_{14}$

Not only *ApNp* is not admitted by the intuitionists, but no alternation at all is admitted by then which does not consists of at least one asserted component. In the ordinary theory of deduction we can prove that *ApCpq* is a thesis, thus:

The premises:

(11.1) *CCpqCCqrCpr*

(18.2) *CNpCpq*

(24.1) *CCpqCCNpqq*

(27.2) *CpApq*

(27.5) *CqApq*

The deduction:

(27.2) $q/Cpq$ *(29.3)

(29.3) *CpApCpq*

(27.5) $q/Cpq$ * (29.4)

(29.4) *CCpqApCpq*

(11.1) $p/Np$, $q/Cpq$, $r/ApCpq$ * C(18.2)–C(29.4)–(29.5)

(29.5) *CNpApCpq*

(24.1) $q/ApCpq$ * C(29.3)–C(29.5)–(29.6)

(29.6) *ApCpq*.

Thesis *ApCpq* means: "*p* or (if *p*, then *q*)". Neither of its components can be asserted, as neither of them is a thesis; *ApCpq* has to be rejected, as shows the matrix $M_{14}$ (take $p/2$, $q/3$). As a matter of fact, (29.6) was proved by help of (24.1) which is rejected by the intuitionists. By adding of (29.6) to the axioms of the intuitionist logic, we get the principle of Peirce *CCCpqpp* from which follows *CCNppp* and the whole theory of deduction.

*ACpqCqp* is another thesis which does not belong to the intuitionist logic, although it is verified by the matrix $M_{14}$ given by Heyting. The matrix of Heyting is not an adequate matrix of the intuitionist logic. A matrix is called "adequate", when it verifies only those theses which are consequences of a given set of axioms verified by it. The adequate matrix of the intuitionist logic is infinite, as was shown by Gödel and Jaśkowski.

The idea that an alternation should be asserted only when at least one of its components is asserted, is in my opinion the most interesting and important feature of the intuitionist logic. Closely connected with this idea is the intuitionist interpretation of existential propositions which, according to the intuitionists, should be built up upon constructions of mathematical facts, and not upon arguments deducing the opposite statements to an absurdity. The whole problem is very difficult, and requires a careful analysis of many ideas, among others of the idea of "assertion".

## D. Conjunction

### 30. The definition of conjunction

I mean by a conjunction the combination of two propositions by means of the word "and". The conjunction "*p* and *q*" is true only when both its components are true, in all the other cases it is false. The following definition by C and N seems not to be very intuitive:

(30.1) *CδNCpNqδKpq*.

The newly–introduced term is *K*, "and". The *definiendum Kpq* means "*p* and *q*". In order to explain the meaning of the *definiens NCpNq*, let us first see what means this expression wthout *N*. *CpNq*, or "if *p*, then not–*q*", means roughly speaking, that *p* and *q* exclude each other not being together true. The negation of *CpNq*, i.e. *NCpNq*, means therefore that *p* and *q* do not exclude each other, and are together true.

There is another more intuitive definition of *Kpq*. For the first *p* in (30.1) we may write *NNp*, as *NNp* means the same as *p*. We have therefore:

(30.2) *CδNCNNpNqδKpq*,

or with respect to the definition (26.1) of *Apq*:

(30.3) *CδNANpNqδKpq*.

According to this definition "*p* and *q*" means the same as "not–(either not–*p* or not–*q*)". The *definiens* is true, when the expression in brackets is false, and this expression is false, when both not–*p* and not–*q* are false, or both *p* and *q* are true. Instead of defining *K* by *A* and *N*, we could define *A* by *K* and *N*, thus:

(30.4) *CδNKNpNqδApq*.

In words: "*p* or *q*["] means the same as "not–(not–*p* and not–*q*)". In symbols:

(30.3) *Kpq* means the same as *NANpNq*,

and

(30.4) *Apq* means the same as *NKNpNq*.

From these definitions and their complements:

(30.5) *CδKpqδNANpNq*

and

(30.6) *CδApqδNKNpNq*,

we get by the substitution δ/' the following, four theses:

(30.7) *CNANpNqKpq*
(30.8) *CKpqNANpNq*
(30.9) *CNKNpNqApq*
(30.10) *CApqNKNpNq*,

which are sometimes called, not rightly, the laws of De Morgan. De Morgan was an outstandirg English logician, but the logical laws discovered by him and called after his name do not belong to the theory of propositions, but to the theory of classes. On the other hand, theses (30.7) till (30.10) were already known to the mediaeval logicians. We read, for instance, in the conmentary of Versorius on the classical textbook of mediaeval logic of Petrus Hispanus that "*Copulativa et disiunctiva de partibus contradicentibus contradicunt*". *Copulativa* means conjunction, "*p* and *q*", *disiunctiva* — alternation, "*p* or *q*". According to this text, "*p* and *q*" and "not–*p* or not–*q*" are contradictory to each ofher, and consequently "*p* and *q*" is equivalent to the negation of "not–*p* or not–*q*". Similarly "*p* or *q*" is equivalent to the negation of "not–*p* and not–*q*". The same idea is expressed more exactly by Ockham: "*Opposita contradictoria disiunctivae est una Copulativa ex contradictoribus partium ipsius disiunctivae*", i.e. the contradictory of "*p* or *q*" is "not–*p* and not–*q*". It is not known to me who first discovered these theses.

### 31. The most important *C–K*–theses

To the most important *C–K*–theses belong the following three:

(31.1) *CKpqp*
(31.2) *CKpqq*
(31.3) *CpCqKpq*.

In words: "If *p* and *q*, then *p*", "if *p* and *q*, then *q*", "if *p*, then (if *q*, then *p* and *q*)". The first two theses are called in *Principia*, like thesis *CpCqp*, the "principles of simplification". I shall derive all the three from some *C*– and *C–K*–theses by help of the definition (30.1).

The premisses:

(10.1) *CpCqp*
(11.9) *CCqrCCpqCpr*
(12.5) *CpCCpqq*
(18.2) *CNpCpq*
(21.2) *CCpNqCqNp*
(21.3) *CCNpqCNqp*
(30.1) *CδNCpNqδKpq*

The deduction:

(18.2) *q*/*Nq* * (31.4)
(31.4) *CNpCpNq*
(18.1) *p*/*Nq*, *q*/*p* * (31.5)
(31.5) *CNqCpNq*
(12.5) *q*/*Nq* * (31.6)
(31.6) *CpCCpNqNq*
(11.9) *q*/*CpNq*, *r*/*CqNp*, *p*/*r* * C(21.2)–(31.7)
(31.7) *CCrCpNqCrCqNp*
(21.3) *q*/*CpNq* * C(31.4)–(31.8)
(31.8) *CNCpNqp*
(21.3) *p*/*q*, *q*/*CpNq* * C(31.5)–(31.9)
(31.9) *CNCpNqq*

(31.7) *r* / *p*, *p* / *CpNq* * C(31.6)–(31.10)

(31.10) *CpCqNCpNq*

(30.1) δ / *C'p* * C(31.8)–(31.1)

(31.1) *CKpqp*

(30.1) δ / *C'q* * C(31.9)–(31.2)

(31.2) *CKpqq*

(30.1) δ / *CpCq'* * C(31.10)–(31.3)

(31.3) *CpCqKpq*

One of my ancient pupils, Sobociński, has proved that the above three *C*–*K*–theses combined with the implicational logic which can be built up on axiom (16.1), are a sufficient basis for all the *C*–*K*–theses. I shall therefore derive the next theses without help of the definition (30.1).

The following thesis was called by Peano, according to *Principia*, the "principle of composition":

(31.14) *CCpqCCprCpKqr*.

In words: "If (if *p*, then *q*), then if (if *p*, then *r*), then (if *p*, then *q* and *r*)". The principle is evidently true. I take as premisses of the proof the theses:

(11.9) *CCqrCCpqCpr*

(13.1) *CCpCqrCCpqCpr*

(31.3) *CpCqKpq*.

The proof:

(31.3) *p* / *q*, *q* / *r* * (31.11)

(31.11) *CqCrKqr*

(11.9) *r* / *CrKqr* * C(31.11)–(31.12)

(31.12) *CCpqCpCrKqr*

(11.9) *q* / *CpCqr*, *r* / *CCpqCpr*, *p* / *s* * C(13.1)–(31.13)

(31.13) *CCsCpCqrCsCCpqCpr*

(31.13) *s* / *Cpq*, *q* / *r*, *r* / *Kqr* * C(31.12)–(31.14)

(31.14) *CCpqCCprCpKqr*.

The next thesis:

(31.15) *CKpqKqp*

is the "commutative law" for conjunction or, as the authors of *Principia* say, for "logical multiplication". By an analogy to algebraic operations conjunction is sometimes called "logical multiplication", as alternation is called "logical addition". As this analogy does not go far enough, it should be rather dropped.

The proof of (31.15) runs:

(31.14) *p* / *Kpq*, *r* / *p* * C(31.2)–C(31.1)–(31.15)

(31.15) *CKpqKqp*.

Closely connected with (31.15) is the next thesis:

(31.18) *CCpqCKprKqr*,

called by Peano, according to *Principia*, the "principle of the factor". It is explained in *Principia* thus: "Both sides of an implication may be multiplied by a common factor". The proof requires as premisses the principle of commutation (12.1) and the theses (31.2) and (31.14):

(31.2) *q* / *r* * (31.16)

(31.16) *CKprr*

(12.1) *p* / *Cpq*, *q* / *Cpr*, *r* / *CpKqr* * C(31.14)–(31.17)

(31.17) *CCprCCpqCpKqr*

(31.17) *p* / *Kpr* * C(31.16)–(31.18)

(31.18) *CCpqCKprKqr*.

The last two theses I want to mention here are most frequently used:

(31.21) *CCpCqrCKpqr*

(31.23) *CCKpqrCpCqr*

In words: "If [if *p*, then (if *q*, then *r*)], then (if *p* and *q*, then *r*)"; "if (if *p* and *q*, then *r*), then [if *p*, then (if *q*, then *r*)]". Both theses are implications which are converse to each other and give together an equivalence. The first was called by Peano, according to *Principia*, the "principle of importation", the second the "principle of exportation". In the first thesis we "import" *q* into the antecedent of *CpCqr* forming *CKpqr*, in the second we "export" *q* from the antecedent of *CKpqr* forming *CpCqr*. Theses (31.21) and (31.23) can be proved from (31.1), (31.2) and (31.3) by the premisses:

(11.1) *CCpqCCqrCpr*

(11.11) *CCqrCCsCpqCsCpr*

(12.1) *CCpCqrCqCpr*

(31.14) *CCsCpCqrCsCCpqCpr*

(11.1) *p* / *Kpq*, *q* / *p*, *r* / *Cqr* * C(31.1)–(31.19)

(31.19) *CCpCqrCKpqCqr*

(31.14) *s* / *CpCqr*, *p* / *Kpq* * C(31.19)–(31.20)

(31.20) *CCpCqrCCKpqqCKpqr*

(12.1) *p* / *CpCqr*, *q* / *CKpqq*, *r* / *CKpqr* * C(31.20)–C(31.2)–(31.21)

(31.21) *CCpCqrCKpqr*

(12.1) *p* / *Cqr*, *q* / *CsCpq*, *r* / *CsCpr* * C(11.11)–(31.22)

(31.22) *CCsCpqCCqrCsCpr*

(31.22) *s* / *p*, *p* / *q*, *q* / *Kpq* * C(31.3)–(31.23)

(31.23) *CCKpqrCpCqr*.

Theses (31.21) and (31.23) are very useful, when we want to reduce the number of the antecedents in expressions of the form $C\alpha_1 C\alpha_2 C\alpha_3 ... C\alpha_n \beta$. In expressions of this form I call $\alpha_1$, the first antecedent, $\alpha_2$ the second antecedent, $\alpha_k$ the *k*–th antecedent. By means of the principles of importation and exportation we easily can transform such expressions into equivalent formulae with fewer antecedents. So is $C\alpha_1 C\alpha_2 C\alpha_3 ... C\alpha_n \beta$ equivalent to $CK\alpha_1 \alpha_2 C\alpha_3 ... C\alpha_n \beta$, this last expression is equivalent to $CKK\alpha_1 \alpha_2 \alpha_3 ... C\alpha_n \beta$, and so on. These equivalences are needed, for instance, for the proof of the theorem that all the antecedents of the expression $C\alpha_1 C\alpha_2 C\alpha_3 ... C\alpha_n \beta$ are interchangeable. It suffices to show for this purpose that any two consecutive antecedents, as $\alpha_{k-1}$ and $\alpha_k$, are interchangeable. For $\alpha_1$ and $\alpha_2$, this can be directly proved by the principle of commutation; for $\alpha_2$ and $\alpha_3$, by the thesis $CC\alpha_1 C\alpha_2 C\alpha_3 \beta C\alpha_1 C\alpha_3 C\alpha_2 \beta$ which arises by the application of the second form of the syllogism to the principle of commutation. In the same way we could form the thesis $CC\alpha_1 C\alpha_2 C\alpha_3 C\alpha_4 \beta C\alpha_1 C\alpha_2 C\alpha_4 C\alpha_3 \beta$, and similar theses for five and six antecedents, but it is impossible to write down such a thesis for *n* antecedents, where *n* is any natural number. Applying theses (31.21) and (31.23) we need for the proof of the above theorem only the principle of commutation and the thesis:

(31.24) *CCsCpCqrCsCqCpr*.

In order to prove that $\alpha_{k-1}$ and $\alpha_k$ are interchangeable, we transforn the expression $C\alpha_1 C\alpha_2 C\alpha_3 ... C\alpha_{k-2} C\alpha_{k-1} C\alpha_k ... C\alpha_n \beta$ by applying thesis (31.21) as many times as necessary into the conjunctional form $CK\gamma\alpha_{k-2}C\alpha_{k-1}C\alpha_k...C\alpha_n\beta$ where $\gamma$ denotes the conjunction of the antecedents $\alpha_1$–$\alpha_{k-3}$; then we apply thesis (31.24) getting the expression $CK\gamma\alpha_{k-2}C\alpha_kC\alpha_{k-1}...C\alpha_n\beta$, and by successive aplications of thesis (31.23) we obtain again the purely implicational form $C\alpha_1C\alpha_2...C\alpha_{k-2}C\alpha_kC\alpha_{k-1}...C\alpha_n\beta$.

Thesis (31.21) is frequently used to transform implicational expressions into conjunctional ones. So we get, for instance, from the syllogism *CCpqCCqrCpr* the form *CKCpqCqrCpr*, and from *CCqrCCpqCpr*, the form *CKCqrCprCpr*. Theses:

(31.25) *CKCpqCqrCpr*

and

(31.26) *CKCqrCpqCpr*

which in *Principia* also are called the "principles of the syllogism", are not so convenient, as their implicational forms (11.1) and (11.9). If two premisses of the form $C\alpha\beta$ and $C\beta\gamma$ are asserted, we obtain immediately by two detachments the conclusion $C\alpha\gamma$ using the implicational forms of the syllogism. When we only have to our disposition the conjunctional forms, we must first prove by means of thesis (31.3) and by two detachments the conjunction $KC\alpha\beta C\beta\gamma$, and then we may obtain the conclusion $C\alpha\gamma$ by a third detachment. Besides, when only $C\alpha\beta$ is asserted, we can deduce froim it by (11.1) the consequence $CC\beta rC\alpha r$, and by (11.9) the consequence $CCp\alpha pCp\beta$, whereas neither of these consequences can be obtained fron the conjunctional forms (31.25) and (31.26).

**32. The principle of excluded contradiction**

According to Aristotle, the most fundamental principle of all is that two contradictory propositions are not together true. This principle runs in symbols:

(32.1) *NKpNp*.

That means literally: "Not–(*p* and not–*p*)". Thesis (32.1) is like the principle of excluded middle not an axiom of our system. The authors of *Principia* have already observed proving the "law of contradiction" that in spite of its fame they have found few occasions for its use. It can be proved, curiously enough, on the basis of the first principle of double negation and the definition of conjunction:

(20.1) *CpNNp*

(30.1) $C\delta NCpNq\delta Kpq$

  (20.1) $p/CpNNp$ * C(20.1)–(32.2)

(32.2) *NNCpNNp*

  (30.1) $\delta/N'$, $q/Np$ * C(32.2)–(32.1)

(32.1) *NKpNp*.

On the basis of definition (30.1) *NKpNp* means the same as *NNCpNNp*. As it is plain that this latter thesis, undetacheable like *NKpNp*, has no important consequences, the principle of excluded contradiction is as axiom of no use whatever.

The importance falsely attributed to this principle belongs in fact to the principle of Duns Scotus. As we know already, if two contradictory propositions were asserted both, we could deduce from them by the principle of Duns Scotus any proposition. Let us transform the first form of this principle:

(18.1) *CpCNpq*

by help of the principle of importation:

(31.21) *CCpCqrCKpqr*

into the conjunctional form:

(31.21) $q/Np,\ r/q$ * C(18.1)–(32.3)

(32.3) $CKpNpq$.

Thesis (32.3) is an exact translation in symbols of the principle set forth by Duns Scotus. A conjunction of two contradictories, like $p$ and not–$p$, is called "*contradictio in forma*". The statement of Duns Scotus runs: *Ad quamlibet propositionem implicantem contradictionem in forma sequitur quaelibet alia propositio in consequentia formali*. The proof of this statement, as given by Duns Scotus and expounded in section 18 in words, can now be translated into symbols:

The supposition reads:

(I) $K\alpha N\alpha$ (*Socrates est et Socrates non est.*)

The premisses are:

(31.1) $CKpqp$

(31.2) $CKpqq$

(27.2) $CpApq$

(26.5) $CApqCNpq$

The proof:

(31.1) $p/\alpha,\ q/N\alpha$ * C(I)–(II)

(II) $\alpha$

(31.2) $p/\alpha,\ q/N\alpha$ * C(II)–(III)

(III) $N\alpha$

(27.2) $p/\alpha$ * C(II)–(IV)

(IV) $A\alpha q$

(26.5) $p/[\alpha]$ [*] C(IV)–C(III)–(V)

(V) $q$ (*Baculus stat in angulo.*)

This proof is correct, but it has one flaw: The supposition $K\alpha N\alpha$ cannot be asserted, because it is false, and to assert a false proposition is an error. This point will be considered later, when we shall come to speak about the so–called "rejection". For the moment it will be instructive to compare the above argument of Duns Scotus with the following flawles proof of the thesis (32.3).

The premisses are the same, as in Duns Scotus' argument, but enforced by the addition of two inplicational theses:

(11.1) $CCpqCCqrCpr$

(13.11) $CCpCpqCpq$.

The proof:

(31.1) $q/Np$ * (32.4)

(32.4) $CKpNpp$

(31.2) $q/Np$ * (32.5)

(32.5) $CKpNpNp$

(11.1) $p/KpNp,\ q/p,\ r/Apq$ * C(32.4)–C(27.2)–(32.6)

(32.6) $CKpNpApq$

(11.1) $p/KpNp,\ q/Apq,\ r/CNpq$ * C(32.6)–C(26.5)–(32.7)

(32.7) $CKpNpCNpq$

(11.1) $p/KpNp,\ q/Np,\ r/q$ * C(32.5)–(32.8)

(32.8) $CCNpqCKpNpq$

(11.1) $p/KpNp,\ q/CNpq,\ r/CKpNpq$ * C(32.7)–C(32.8)–(32.9)

(32.9) $CKpNpCKpNpq$

(13.11) $p/KpNp$ * C(32.9)–(32.3)

(32.3) *CKpNpq*.

Thesis (32.3) can be obtained directly from the second form of the principle of Duns Scotus and the principle of contradiction:

(18.2) *CNpCpq*

(32.1) *NKpNp*

(18.2) *p* / *KpNp* * C(32.1)–(32.3)

(32.3) *CKpNpq*.

The antecedent of thesis (32.3), *KpNp*, is obviously a false proposition, and entails therefore any proposition according to the principle: *ad falsum sequitur quodlibet*. This scholastic principle may contribute to the explanation, why so much importance is still attached to the principle of contradiction in spite of its uselessness as axiom. Every author of a deductive system is bound to prove that his system is consistent. Consistency is frequently mixed up with non–contradiction; everybody therefore in anxious to prove that his system does not imply two contradictory theses against the principle of excluded contradiction. This principle seems thus to play an important role in construction of deductive systems. It is true, in fact, that a consistent system is not contradictory, and a contradictory system is not consistent. But non–contradiction is a wider idea than consistency, and therefore not identical with it. There are not contradictory systems which are inconsistent. Take for example a system of C–theses (without *N*) based on the following two axioms:

(I) *CpCqq*

(II) *CCpqCqp*.

It follows from these axioms:

(I) *p* / *CpCqq* * C(I)–(III)

(III) *Cqq*

(II) *q* / *Cqq* * C(I)–C(III)–(IV)

(IV) *p*.

This system leads to a variable, like a contradictory system, and includes therefore all significant C–expressions, among them the false ones too. Nevertheless it is not contradictory, for it is impossible to construct in it two propositions of the form $\alpha$ and $N\alpha$. *N* does not exists in it. Non–contradiction as criterion of consistency can be applied only to systems with negation. What we want, however, when we are constructing a deductive system, is to exclude from it all false propositions, and therefore we are so anxious to exclude contradictories, because one of two contradictories must always be false. But the exclusion of contradictories is not sufficient, as false theses may arise in a system not only by accepting contradictory propositions. It was therefore a lucky idea of the American logician E. Post to define a system as consistent which does not include all its significant expressions. This definition of consistency is wider than the usual definition by non–contradiction, and can be applied to all deductive systems.

Aristotie discusses at length the principle of excluded contradiction in the famous Book Γ of his Metaphysics. The tremendous influence of this discussion is, in my opinion, the reason that this principle was proclaimed as the most fundamental basis of the human thught. It is a true principle of our two–valued logic, but it is far less important than the principle of Duns Scotus or some other axioms of this logic. We shall see later that by a slight modification of the functor *N* it will be possible to construct a many–valued system of logic in which two centradictories may be toqether true whereas the system is consistent in the sense of Post.

**33.** ***K–N*–theses and the intuitionist logic**

The functors *K* nad *N*, like *A* and *N*, form a sufficiert semantic basis for the theory of deduction, i.e. all the other functors of this theory can be defined by them. It suffices for this purpose te show that *C* may be defined by *K* and *N*:

(33.1) *CδNKpNqδCpq*.

According to this definition, "if *p*, then *q*" means the same as "not–(*p* and not–*q*)". It seems that this evident connexion betweeen the functors *K–N* and *C* was known to the Stoics, because we read in Cicero's *De fato* that according to Chrysippus the sentences: "*si quis oriente Canicula natus est, is in mari non morietur*" and "*non: et natus est is oriente Canicula et in mari morietur*", are equivalent. By means of definition (33.1) all the *C–N*–theses can be translated into *K–N*–theses. The system *K–N* was axiomatized by Sobociński.

Gödel observed that all the *K–N*–theses are true in the intuitionist logic. As we already know, the intuitionist logic is only a proper part of the theory of deduction: it can be based, as regards the *C*–theses, on the positive logic, i.e. on the principle of simplification and the principle of Frege, and as regards the *C–N*–theses, on the principle of Duns Scotus and either on the principle of destructive dillema *CCpqCCpNqNp*, or the second principle of transposition *CCpNqCqNp*. The intuitionists accept besides as *C–A*–axioms the theses *CpApq*, *CqApq* and *CCprCCqrCApqr*, and as *C–K*–axioms the theses *CKpqp*, *CKpqq* and *CpCqKpq* (or *CCpqCCprCpKqr*). The functors *C*, *N*, *A* and *K* are all primitive terms in the intuitionist logic, because none of them can be defined by the others. The intuitionists reject the principie cf Peirce *CCCpqpp*, the principle of Clavius *CCNppp*, the second principle of double negation *CNNpp*, the fourth principle of transposition *CCNpNqCqp*, and many other theses. The intuitionist logic being a part of the ordinary system, is weaker than the theory of deduction. I shall prove, however, this somewhat paradoxical theorem that the intuitionist logic, althought weaker than the ordinary system includes as its consequence a system isomorphic to the whole theory of deduction.

As premisses of my proof I accept the following axioms of the intuitionist logic (I omit the *C–A*–axioms and the principle of Duns Scotus, as irrelevant for the proof):

| | |
|---|---|
| (33.2) *CpCqp* | (10.1) |
| (33.3) *CCpCqrCCpqCpr* | (13.1) |
| (33.4) *CKpqp* | (31.1) |
| (33.5) *CKpqq* | (31.2) |
| (33.6) *CpCqKpq* | (31.3) |
| (33.7) *CCpNqCqNp* | (21.2) |

From (33.2) and (33.3) follow the theses:

| | |
|---|---|
| (33.8) *Cpp* | (9.1) |
| (33.9) *CpCCpqq* | (12.5) |
| (33.10) *CCpCqrCqCpr* | (12.1) |
| (33.11) *CCpqCCqrCpr* | (11.1) |
| (33.12) *CCqrCCpqCpr*, | (11.9) |

which can be easily proved on the ground of deductions expounded in section 13. They belong all to positive, and therefore to intuitionist logic. From the axioms (33.4), (33.5) and (33.6) we get the theses:

| | |
|---|---|
| (33.13) *CCpqCCprCpKqr* | (31.14) |
| (33.14) *CKpqKqp* | (31.15) |

(33.15) *CCpCqrCKpqr* (31.21)
(33.16) *CCKpqrCpCqr*, (31.23)

as it was shown in section 31 by help of theses (11.1), (11.9), (11.11), (12.1) and (13.1) which all belong to positive logic. I admit theses (33.8)–(33.16) without repeating their proofs. The deduction which I give below consists of three parts: (A) I shall prove the theses *CCpqNKNq* and *CCpqNKNqp*, (B) the theses *NKNKppp* and *NKpNNKNpq*, (C) the thesis *NKNKpNqNNKNKpqNNKpr*.

(A)

(33.8) *p* / *Nq* * (33.17)
(33.17) *CNqNq*
(33.7) *p* / *Nq* * C(33.17)–(33.18)
(33.18) *CqNNq*
(33.7) *p* / *Nq* * (33.19)
(33.19) *CCpNNqCNqNp*
(33.12) *r* / *NNq* * C(33.18)–(33.20)
(33. 20) *CCpqCpNNq*
(33.11) *p* / *Cpq*, *q* / *CpNNq*, *r* / *CNqNp* * C(33.20)–C(33.19)–(33.21)
(33.21) *CCpqCNqNp*
(33.12) *q* / *Cpq*, *r* / *CNqNp*, *p* / *r* * C(33.21)–(33.22)
(33.22) *CCrCpqCrCNqNp*
(33.22) *r* / *p*, *p* / *Cpq* * C(33.9)–(33.23)
(33.23) *CpCNqNCpq*
(33.15) *q* / *Nq*, *r* / *NCpq* * C(33.23)–(33.24)
(33.24) *CKpNqNCpq*
(33.7) *p* / *KpNq*, *q* / *Cpq* * C(33.24)–(33.25)
(33.25) *CCpqNKpNq*
(33.14) *p* / *Nq*, *q* / *p* * (33.26)
(33.26) *CKNqpKpNq*
(33.11) *p* / *KNqp*, *q* / *KpNq*, *r* / *NCpq* * C(33.26)–C(33.24)–(33.27)
(33.27) *CKNqpNCpq*
(33.7) *p* / *KNqp*, *q* / *Cpq* * C(33.27)–(33.28)
(33.28) *CCpqNKNqp*.

(B)

(33.13) *q* / *p*, *r* / *p* * C(33.8)–C(33.8)–(33.29)
(33.29) *CpKpp*
(33.28) *q* / *Kpp* * C(33.29)–(33.30)
(33.30) *NKNKppp*
(33.4) *p* / *Np* * (33.31)
(33.31) *CKKpqNp*
(33.7) *p* / *KNpq*, *q* / *p* * C(33.31)–(33.32)
(33.32) *CpNKNpq*
(33.25) *q* / *NKNpq* * C(33.32)–(33.33)
(33.33) *NKpNNNKNpq*.

(C)

(33.22) *r* / *Cpq*, *p* / *Nq*, *q* / *Np* * C(33.21)–(33.34)
(33.34) *CCpqCNNpNNq*

(33.12) *q* / *Cpq*, *r* / *CNNpNNq*, *p* / *r* * C(33.34)–(33.35)
(33.35) *CCrCpqCrCNNpNNq*
(33.35) *r* / *p*, *p* / *q*, *q* / *Kpq* * C(33.6)–(33.36)
(33.36) *CpCMqNNKpq*
(33.10) *q* / *NNq*, *r* / *NNKpq* * C(33.36)–(33.37)***
(33.37) *CNNqCpNNKpq*
(33.12) *q* / *CpNq*, *r* / *CqNp*, *p* / *r* * C(33.37)–(33.38)
(33.38) *CCrCpNqCrCqNp*
(33.38) *r* / *KNq*, *q* / *NKpq* * C(33.37)–(33.39)
(33.39) *CNNqCNKpqNp*
(33.22) *r* / *NNq*, *p* / *NKpq*, *q* / *Np* * C(33.39)–(33.40)
(33.40 ) *CNNqCKNNpNNKpq*
(33.10) *p* / *NNq*, *q* / *NNp*, *r* / *NKpq* * C(33.40)–(33.41)
(33.41) *CNNpCNNqNNKpq*
(33.15) *p* / *NNp*, *q* / *NNq*, *r* / *NNKpq* * C(33.41)–(33.42)
(33.42) *CKNNpNNqNNKpq.*

*

(33.34) *p* / *Kpq* * C(33.5)–(33.43)
(33.43) *CNNKpqNNq*
(33.11) *p* / *Kpq* * C(33.5)–(33.44)
(33.44) *CCqrCKpqr*
(33.44) *q* / *NKpq*, *r* / *NNq*, *p* / *NNr* * C(33.43)–(33.45)
(33.45) *CKNNrNNKpqNNq*
(33.4) *p* / *NNr*, *q* / *NNKpq* * (33.46)
(33.46) *CKNNrNNKpqNNr*
(33.13) *p* / *KNNrNNKpq*, *q* / *NNr*, *r* / *NNq* *
C(33.46)–C(33.45)–(33.47)
(33.47) *CKNNrNNKpqKNNrNNq*
(33.42) *p* / *r* * (33.48)
(33.48) *CKNNrNNqNNKrq*
(33.11) *p* / *KNNrNNKpq*, *q* / *KNNrNNq*, *r* / *NNKrq* *
C(33.47)–C(33.43)–(33.49)
(33.49) *CKNNrKNKpqNNKrq*
(33.16) *p* / *NNr*, *q* / *NNKpq*, *r* / *NNKrq* * C(33.49)–(33.50)
(33.50) *CNNrCNNKpqNNKrq*
(33.10) *p* / *NNr*, *q* / *NNKpq*, *r* / *NNKrq* * C(33.50)–(33.51)
(33.51) *CNNKpqCNNrNNKrq*
(33.38) *r* / *NNKKpq*, *p* / *NNr*, *q* / *NKrq* * C(33.51)–(33.52)
(33.52) *CNNKpqCNKrqNNNr*
(33.10) *p* / *NNKpq*, *q* / *NKrq*, *r* / *NNNr* * C(33.52)–(33.53)
(33.53) *CNKrqCNNKpqNNNr*
(33.15) *p* / *NKrq*, *q* / *NNKpq*, *r* / *NNNr* * C(33.53)–(33.54)
(33.54) *CKNKrqNNKpqNNNr*
(33.34) *p* / *Kpq*, *q* / *p* * C(33.4)–(33.55)
(33.55) *CNNKpqNNp*
(33.44) *q* / *NNKpq*, *r* / *NNp*, *p* / *NKrq* * C(33.55)–(33.56)
(33.56) *CKNKrqNNKpqNNp*

(33.13) *p* / *KNKrqNNKpq*, *q* / *NNp*, *r* / *NNNr* *
C(33.56)–C(33.54)–(33.57)

(33.57) *CKNKrqNNKpqKNNpNNNr*

(33.42) *p* / *Nr* * (33.58)

(33.58) *CKNNpNNNrNNKpNr*

(33.11) *p* / *KNKrqNNKpq*, *q* / *KNNpNNNr*, *r* / *NNKpNr* *
C(33.57)–C(33.58)–(33.59)

(33.59) *CKNKrqNNKpqNNKpNr*

(33.7) *p* / *KNKrqNNKpq*, *q* / *NKpNr* * C(33.59)–(33.60)

(33.60) *CNNpNrNKNKrqNNKpq*

(33.60) *r* / *q*, *q* / *r* * (33.61)

(33.61) *CNKpNqNKNKqrNNKpr*

(33.25) *p* / *NKpNq*, *q* / *NKNKqrNNKpr* * C(33.61)–(33.62)

(33.62) *NKNKpNqNNKNKqrNNKpr*

The three *K*–*N*–theses we got by this deduction, (33.30), (33.33) and (33.62), are too general for our purpose. The special cases needed for our proof can be derived from them by replacing some variables by their negations:

(33.30) *p* / *Np* * (33.63)

(33.63) *NKNKNpNpNp*

(33.33) *p* / *Nq* * (33.64)

(33.64) *NKpNNKNpNq*

(33.62) *r* / *Nr* * (33.65)

(33.65) *NKNKpNqNNKNKqNrNNKpNr.*

Let us now introduce a definition for expressions of the form *NK*α*N*β. We cannot use for this purpose the definition:

(33.1) *C*δ*NKpNq*δ*Cpq*,

because a consequence of this definition:

(33.66) *CNKpNqCpq*,

does not belong to intuitionist logic. The matrix $M_{14}$ of Heyting given in section 29 for *C*, *N* and *A*, and cmmpleted by *K*, as shown in $M_{14.1}$, verifies all the axioms of the intuitionist logic (and some other theses not belonging to this logic), but does not verify (33.66), because we have for *p* / 1, *q* / 2: *CNK*1*N*2*C*12 = *CNK*132 = *CN*32 = *C*12 = 2.

| C | 1 | 2 | 3 | *N* |
|---|---|---|---|---|
| *1 | 1 | 2 | 3 | 3 |
| 2 | 1 | 1 | 3 | 3 |
| 3 | 1 | 1 | 1 | 1 |

| *A* | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 |
| 3 | 1 | 2 | 3 |

| K | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 |

$M_{14.1}$

Nothing, however, can prevent us from introducing for sake of abbreviation an entirely new functor, say *G*, and state the following definition:

(33.67) *CδNKpNqδGpq*.

Applying this definition to theses (33.63), (33.64) and (33.65) we have:

(33.67) δ / *NK'Np*, *p* / *Np*, *q* / *p* * C(33.63)–(33.68)

(33.68) *NKGNppNp*

(33.67) δ / ', *p* / *GNpp*, *q* / *p* * C(33.68)–(33.69)

(33.69) *GGNppp*

(33.67) δ / *NKpN'*, *p* / *Np* * C(33.64)–(33.70)

(33.70) *NKpNGNpq*

(33.67) δ / ', *q* / *GNpq* * C(33.70)–(33.71)

(33.71) *GpGNpq*

(33.67) δ / *NK'NNKNKqNrNNKpNr* * C(33.65)–(33.72)

(33.72) *NKGpqNNKNKqNrNNKpNr*

(33.67) δ / *NKGpqNNK'NNKpNNpNr*, *r* / *q*, *q* / *r* * C(33.72)–(33.73)

(33.73) *NKGpqNNKGqrNNKpNr*

(33.67) δ / *NKGpqNNKGqrN'*, *q* / *r* * C(33.73)–(33.74)

(33.74) *NKGpqNNKGqrNGpr*

(33.67) δ / *NKGpqN'*, *p* / *Gqr*, *q* / *Gpr* * C(33.74)–(33.75)

(33.75) *NKGpqNGGqrGpr*

(33.67) δ / ', *p* / *Gpq*, *q* / *GGqrGpr* * C(33.75)–(33.76)

(33.76) *GGpqGGqrGpr*.

Let us now compare theses (33.76), (33.71) and (33.69) with the set of axioms (11.1), (18.l) and (19.1) which I mentioned in section 19 as a sufficient basis of the whole theory of deduction:

| | |
|---|---|
| (11.1) *CCpqCCqrCpr* | (33.76) *GGpqGGqrGpr* |
| (18.1) *CpCNpq* | (33.71) *GpGNpq* |
| (19.1) *CCNppp* | (33.69) *GGNppp* |

Both sets are essentially identical, because it makes no difference whether we denote the implicational functor by C or by G. To show therefore that the set with G is a sufficient basis for the whole theory of deduction, we only have to prove that the rule of detachment valid for C is also valid for G. This cannot be done immediately by a thesis of the intuitionistic logic, but requires a special consideration. Thesis *CNKpNqCpq* or *CGpqCpq* which would serve our purpose is not true in our system. We can prove, however, a similar thesis, *CNKpNqCpNNq*:

(33.11) *p* / *q*, *q* / *Npq*, *r* / *CpNNKpq* * C(33.18)–C(33.37)–(33.77)

(33.77) *CqCpNNKpq*

(33.10) *p* / *q*, *q* / *p*, *r* / *NNKpq* * C(33.77)–(33.78)

(33.78) *CpCqNNKpq*

(33.38) *r* / *p*, *p* / *q*, *q* / *NKpq* * C(33.78)–(33.79)

(33.79) *CpCNKpqNq*

(33.10) $q/NKpq$, $r/Nq$ * C(33.79)–(33.80)

(33.80) *CNKpqCpNq*

(33.80) $q/Nq$ * (33.81)

(33.81) *CNKpNqCpNNq*.

From (33.81) we obtain further consequences taking as premiss thesis *CCqrCCsCpqCsCpr* which results from the second form of the syllogism and belongs to positive logic.

(33.82) *CCqrCCsCpqCsCpr*

(33.81) $q/Nq$ * (33.83)

(33.83) *CNKpNNqCpNNNq*

(33.21) $p/q$, $q/NNq$ * C(33.18)–(33.84)

(33.84) *CNNNqNq*

(33.32) $q/NNNq$, $r/Nq$, $s/NKNNq$ * C(33.84)–C(33.83)–(33.85)

(33.85) *CNKpNNqCpNq*

(33.85) $q/KqNr$ * (33.86)

(33.86) *CNKpNNKqNrCpNKqNr*

Let us now apply the definition (33.67) to the theses (33.81), (33.85) and (33.86):

(33.67) $\delta/C'CpNNq$ * C(33.81)–(33.87)

(33.87) *CGpqCpNNq*

(33.67) $\delta/C'CpNq$, $q/Nq$ * C(33.85)–(33.88)

(33.88) *CGpNqCpNq*

(33.67) $\delta/CNKpN'Cp'$, $p/q$, $q/r$ * C(33.86)–(33.89)

(33.89) *CNKpNGqCpGqr*

(33.67) $\delta/C'CpGqr$, $q/Gqr$ * C(33.89)–(33.90)

(33.90) *CGpGqrCpGqr*.

Supposing that both $G\alpha\beta$ and $\alpha$ are asserted we get from (33.81) by two detachmnents the conclusion $NN\beta$. In a similar way, if $G\alpha N\beta$ and $\alpha$ are asserted, we get from (33.88) as conclusion $N\beta$ and if $G\alpha G\beta\gamma$ and $\alpha$ are asserted, we get from (33.90) $G\beta\gamma$. Three rules of inference are thereby established:

(a) $G\alpha\beta, \alpha \rightarrow NN\beta$

(b) $G\alpha N\beta \rightarrow N\beta$

(c) $G\alpha G\beta\gamma \rightarrow G\beta\gamma$

In the *G–N*–system every expression is either a variable, or a negation beginning with *N*, or an implication beginning with *G*. If $\beta$ is a variable, say *p*, we get by (a) from $G\alpha p$ and $\alpha$ the conclusion *NNp*, and by substitution *NNNp*. As the principle of Duns Scotus is admitted in our system, we obtain from *CNNpCNNNpp* by two detachments *p*. Therefore, when $\beta$ is a variable, the rule of detachment:

(d) $G\alpha\beta, \alpha \rightarrow \beta$

is valid. It is plain, according to (b) and (c), that the same rule is valid too, when $\beta$ is a negation or an implication. Consequently the rule of detachment (d), like that for *C*, is valid in all cases. The proof that the intuitionistic system contains the whole theory of deduction as its proper part is thus completed.

## (E) Equivalence

### 34. The definition of equvalence

Equivalence is a conjunction of two implications which are converse to each other. The most intuitive definition of equivalence is the following:

(34.1) *CδKCpqCqpαβγEpq.*

The new introduced term is *E*. *Epq* reads in words "*p* when and only when *q*", and means the same as "(if *p*, then *q*) and (if *q*, then *p*)".

We also may define equivalence by our primitive terms, *C* and *N*, replacing *K* according to the definition (30.1). This other definition is not so intuitive as the former one, and runs:

(34.2) *CδNCCpqNCqpδEpq.*

Together with the implication (34.1) is given its converse implication, as it was explained in section 25:

(34.3) *CδEpqδKCpqCqp.*

From (34.1) and (34.3) we now derive theses (34.5), (34.8) and (34.10) by help of the premisses:

(11.1) *CCpqCCqrCpr*

(31.1) *CKpqp*

(31.2) *CKpqq*

(31.23) *CCKpqrCpCqr*

(34.1) δ/′ * (34.4)

(34.4) *CKCpqCqpEpq*

(31.23) *p*/*Cpq*, *q*/*Cqp*, *r*/*Epq* * C(34.4)–(34.5)

(34.5) *CCpqCCqpEpq*

(34.3) δ/′ * (34.6)

(34.6) *CEpqKCpqCqp*

(31.1) *p*/*Cpq*, *q*/*Cqp* * (34.7)

(34.7) *CKCpqCqpCpq*

(11.1) *p*/*Epq*, *q*/*KCpqCqp*, *r*/*Cpq* * C(34.6)–C(34.7)–(34.8)

(34.8) *CEpqCpq*

(31.2) *p*/*Cpq*, *q*/*Cqp* * (34.9)

(34.9) *CKCpqCqpCqp*

(11.1) *p*/*Epq*, *q*/*KCpqCqp*, *r*/*Cqp* * C(34.6)–C(34.9)–(34.10)

(34.10) *CEpqCqp.*

Theses (34.5), (34.8) and (34.10) are deductively equivalent to the definition (34.1). We shall see that by introducing variable functors and quantifiers into the theory of deduction we shall be able to define the equivalence by the implication and the universal quantifier without conjunction or negation.

Equivalence shares with the relation of identity and equality the properties of being reflexive, symetric and transitive. The proof requires the following premisses:

(9.1) *Cpp*

(11.1) *CCpqCCqrCpr*

(11.4) *CCpCqrCCsqCpCsr*

(12.1) *CCpCqrCqCpr*

(13.1) *CCpCqrCCpqCpr*

(31.21) *CCpCqrCKpqr*

(34.5) *q/p* * C(9.1)–C(9.1)–(34.11)

(34.11) *Epp* Equivalence is reflexive.

(34.5) *p / q, q / p* * (34.12)

(34.12) *CCqpCCpqEqp*

(11.1) *p / Epq, q / Cqp, r / CCpqEqp* * C(34.10)–C(34.12)–(34.13)

(34.13) *CEpqCCpqEqp*

(13.1) *p / Epq, q / Cpq, r / Eqp* * C(34.13)–C(34.8)–(34.14)

(34.14) *CEpqEqp* Equivalence is symmetric.

(11.1) *p / Epq, q / Cpq, r / CCqrCpr* * C(34.8)–C(11.1)–(34.15)

(34.15) *CEpqCCqrCpr*

(34.8) *p / q, q / r* * (34.16)

(34.16) *CEqrCqr*

(11.4) *p / Epq, q / Cqr, r / Cpr, s / Eqr* * C(34.15)–C(34.16)–(34.17)

(34.17) *CEpqCEqrCpr*

(34.17) *p / r, r / p* * (34.18)

(34.18) *CErqCEqpCrp*

(11.4) *p / Erq, q / Eqp, r / Crp, s / Epq* * C(34.13)–C(34.14)–(34.19)

(34.19) *CErqCEpqCrp*

(34.14) *p / q, q / r* * (34.20)

(34.20) *CEqrErq*

(11.1) *p / Eqr, q / Erq, r / CEpqCrp* * C(34.20)–C(34.19)–(34.21)

(34.21) *CEqrCEpqCrp*

(12.1) *p / Eqr, q / Epq, r / Crp* * C(34.21)–(34.22)

(34.22) *CEpqCEqrCrp*

(31.21) *p / Epq, q / Eqr, r / Crp* * C(34.22)–(34.23)

(34.23) *CKEpqEqrCrp*

(31.21) *p / Epq, q / Eqr, r / Cpr* * C(34.17)–(34.24)

(34.24) *CKEpqEqrCpr*

(34.5) *q / r* * (34.25)

(34.25) *CCprCCrpEpr*

(11.1) *p / KEpqEqr, q / Cpr, r / CCrpEpr* * C(34.24)–C(34.25)–(34.26)

(34.26) *CKEpqEqrCCrpEpr*

(13.1) *p / KEpqEqr, q / Crp, r / Epr* * C(34.26)–C(34.25)–(34.27)

(34.27) *CKEpqEqrEpr*

**35. Some *E*–theses**

A great number of implicational theses can be formulated as equivalences, for instance the principle of commutation, of double negation or of transposition:

(35.1) *ECpCqrCqCpr*

(35.2) *EpNNp*

(35.3) *ECpqCNqNp*

The proofs are easy by applying thesis (34.5) to (12.1) and its substitution *p / q, q / p* in the first case, to (20.1) and (20.2) in the second case, and two (21.1) and (21.4) with the substitution *p / q, q / p* in the third case. All these theses are of little importance, as for proving new propositions we need implications, not equivalences. More interesting are theses with the *E* in the middle, like the following one:

(35.4) *NEpNp*.

According to this thesis no proposition is equivalent to its contradictory. The proof requires eight theses as premisses:

(11.1) *CCpqCCqrCpr*

(19.1) *CCNppp*

(19.2) *CCpNpNp*

(21.1) *CCpqCNqNp*

(31.14) *CCpqCCprCpKqr*

(32.5) *NKpNp*

(34.8) *CEpqCpq*

(34.10) *CEpqCqp*

The proof:

(34.3) *q / Np* * (35.5)

(35.5) *CEpNpCpNp*

(34.10) *q / Np* * (35.6)

(35.6) *CEpNpCNpp*

(11.1) *p / EpNp, q / CpNp, r / Np* * C(35.5)–C(19.2)–(35.7)

(35.7) *CEpNpNp*

(11.1) *p / ErNp, q / CNpp, r / p* * C(35.6)–C(19.1)–(35.8)

(35.8) *CEpNpp*

(31.14) *p / EpNp, q / p, r / Np* * C(35.8)–C(35.7)–(35.9)

(35.9) *CEpNpKpNp*

(21.1) *p / EpNp, q / KpNp* * C(35.9)–C(32.l)–(35.4)

(35.4) *NEpNp*.

If we agree that it is absurd to assert the equivalence of two contradictory propositions, we may call thesis (34.5) the principle of excluded absurdity. Most of the so called "antinomies" are faulty arguments leading to an absurdity. Take for instance the formulation of the famous antinomy of classes given in the *Principia mathematica* (Vol. I, p. 63): "Let *w* be the class of all those classes which are not members of themselves. Then, whatever class *x* may be, "*x* is a *w*" is equivalent to "*x* is not an *x*". Hence, giving to *x* the value *w*, "*w* is a *w*" is equivalent to "*w* is not a *w*"." The statements "*w* is a *w*" and "*w* is not a *w*" are two contradictory propositions, their equivalence, therefore, is an absurdity and must be rejected according to thesis (35.4). The argument leading to this absurdity is faulty, because the definition of the class

"*x* is a *v*" is equivalent to "*x* is not an *x*",

is wrong. According to Leśniewski expression is of the type "*x* is a *w*" can be defined only under the condition that *x* is an object, and *x* is an object when and only when it is true that "*x* is a *x*". The right definition of the class *w* must therefore run:

"*x* is a *w*" is equivalent to "*x* is an *x* and *x* is not an *x*", By giving to *x* the value w get the formula:

"*w* is a *w*" is equivalent to "*w* is a *w* and *w* is not a *w*", which simply means that the proposition "*w* is a *w*" is false. The antinomy disappears and there is no need to introduce in this case a theory of types.

The principle of excluded absurdity is in stronger sense stronger than the principle of excluded contradiction. This can be seen by another *E*–thesis. From

(34.5) *CCpqCCqpEpq*

We get by the premisses:

(12.1) *CCpCqrCqCpr*

(13.8) *CCCqprCpr*
(31.21) *CCpCqrCKpqr*
the following consequences:
(13.8) $q/p$, $p/q$, $r/CCqpEpq$ * C(34.5)–(35.10)
(35.10) *CqCCqpEpq*
(12.1) $p/q$, $q/Cqp$, $r/Epq$ *C(35.10)–(35.11)
(35.11) *CCqpCqEpq*
(13.8) $r/CqEpq$ * C(35.11)–(35.12)
(35.12) *CpCqEpq*
(31.31) $r/Epr$ * C(35.12)–(35.13)
(35.13) *CKpqEpq*.

From *CKpqEpq* follows by substitution *CKpNpEpNp*, and from this latter thesis by transposition *CNEpNpNKpNp*. The principle of excluded contradiction can be thus proved by the principle of excluded absurdity, but conversely the second principle cannot be proved in a similar way by the first, because the converse implication to (35.13), i.e. *CEpqKpq*, does not hold. This can be seen by the simplest matrices verifying all *N*– and *E*–theses: for $p/2$, $q/2$ we get by $M_0$: *CE22K22* = C12 = 2.

| *K* | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 2 |

| *E* | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 1 |

$M_0$

**36. The *E*–system**

The equivalence shares with the implication two important properties: (a) There exists for *E* an analogous rule of detachment, as for *C*. (b) There exist theses in which *E* occurs as the sole functor, whereas no thesis can be built up by *A* or *K* alone. These two properties enable us to construct an *E*–system analogous to the *C*–system.

The rule of detachment for the *E*–system runs: If $E\alpha\beta$ and $\alpha$ are asserted, then $\beta$ must be asserted too. It can be proved as follows: Let us suppose that $E\alpha\beta$ and $\alpha$ are asserted:

I $E\alpha\beta$
II $\alpha$

we get from them by thesis:

(34.8) *CEpqCpq*
(34.8) $p/[\alpha]$, $q/[\beta]$ * CI–CII–III
III $\beta$.

The simplest *E*–thesis is the principle of identity:

(36.1) *Epp*,

which follows from *Cpp* by thesis (34.5). Another thesis is the principle of commutation for *E*:

(36.2) *EEpqEqp*,

which easily can be proved by (34.14). The above two are the only theses cited in the

*Principia mathematica.* I observed, however, about 30 years ago that equivalence is not only reflexive and commutative, but also associative, because by the matrix $M_0$ given in the preceding section the following thesis can be verified:

(36.3) *EEEpqrEpEqr.*

It was plain that the number of *E*–theses is infinite and that they form a system which should be axiomatized.

The first set of axioms for the *E*–system vas given by Leśniewski in 1929. It consisted of the two following axioms:

(36.4) *EEEprEqpErq*

(36.5) *EEpEqrEEpqr.*

Leśniewski proved that all the *E*–theses can be deduced from these axioms by substitution and detachment. Wajsberg discovered in 1932 simpler sets of axioms, and among them two unique axioms, each containing 15 letters, on which the *E*–system may be established. One of these unique axioms runs:

(36.6) *EEEEpqrsEsEpEqr.*

Some other unique axioms were later discovered by Sobociński and myself all containing 15 letters, until I was able to show in 1933 that the *E*–system can be built up on only one of the following three shortest axioms each containing 11 letters:

(36.7) *EEpqEErqEpr*

(36.8) *EEpqEEprErq*

(36.9) *EEpqEErpEqr.*

As what is called the "proof of completeness" is for the *E*–system the shortest one, I shall give it in what follows, deducing first from thesis (36.7) all those consequences which are required for the proof.

(36.7) *EEpqEErqEpr*

(36.7) *p / Epq, q / EErqEpr, r / s* * *E*(36.7)–(36.10)

(36.10) *EEsEErqEprEEpqs*

(36.10) *s / Epq* * *E*(36.7)–(36.11)

(36.11) *EEpqEpq*

(36.7) *p / Epq, q / Epq* * *E*(36.11)–(36.12)

(36.12) *EErEpqEEpqr*

(36.12) *r / Epq, p / Erq, q / Epr* * *E*(36.7)–(36.13)

(36.13) *EEErqEprEpq*

(36.11) *q / p* * (36.14)

(36.14) *EEppEpp*

(36.13) *r / p, q / p* * *E*(36.14)–(36.15)

(36.15) *Epp*

(36.15) *p / q* * (36.16)

(36.16) *Eqq*

(36.7) *p / q, r / p* * *E*(36.16)–(36.17)

(36.17) *EEpqEqp*

(36.7) *p / Epq, q / Eqp* * *E*(36.17)–(36.18)

(36.18) *EErEqpEEpqr*

(36.17) *p / ErEqp, q / EEpqr* * *E*(36.18)–(36.19)

(36.19) *EEEpqrErEqp*

(36.19) *q / r, r / Epq* * (36.20)

(36.20) *EEEprEpqEEpqErp*

(36.10) *s* / *EEprEpq*, *r* / *p*, *p* / *r* * E(36.20)–(36.21)
(36.21) *EErqEEprEpq*
(36.21) *r* / *Epq* * (36.)
(36.22) *EEEpqqEEpEpqEpq*
(36.13) *r* / *Epq*, *p* / *EpEpq* * *E*(36.22)–(36.23)
(36.23) *EEpEpqq*
(36.17) *p* / *EpEpq* * E(36.23)–(36.24)
(36.24) *EqEpEpq*
(36.7) *p* / *EpEpq* * E(36.23)–(36.25)
(36.25) *EErqEEpEpqr*
(36.25) *r* / *Epq*, *q* / *r*, *p* / *q* * (36.26)
(36.26) *EEEpqrEEqEqrEpq*
(36.10) *s* / *EEpqr*, *r* / *q*, *q* / *Eqr* * E(36.26)–(36.27)
(36.27) *EEpEqrEEpqr* (36.5)
(36.17) *p* / *EpEqr*, *q* / *EEpqr* * *E*(36.27)–(36.28)
(36.28) *EEEpqrEpEqr*
(36.27) *p* / *Epq*, *r* / *p* * E(36.17)–(36.29)
(36.29) *EEEpqqp*
(36.19) *p* / *r*, *r* / *p* * (36.30)
(36.30) *EEErqpEpEqr*
(36.19) *p* / *Erq*, *q* / *p*, *r* / *EpEqr* * E(36.30)–(36.31)
(36.31) *EEpEqrEpErq*
(36.13) *r* / *p*, *q* / *r*, *p* / *q* * (36.32)
(36.32) *EEEprEqpEqr*
(36.31) *p* / *EEprEqp* * E(36.32)–(36.33)
(36.33) *EEEprEqpErq* (36.4)
(36.31) *p* / *EEpqr*, *q* / *r*, *r* / *Eqp* * E(36.19)–(36.34)
(36.34) *EEEpqrEEqpr*
(36.34) *p* / *q*, *q* / *r*, *r* / *s* * (36.35)

## Notes

* In the typescript, the title of this section is: "Part I. Theory of Deduction. Chapter I. A Survey of Theses" [AB&JJJ].

** In fact, this exposition does not exist [AB&JJJ].

*** In the typescript, the formula '(33.35) *r*/*NNq*, *q*/*NNKpq* * C(33.37)–(33.40)' was inserted by Łukasiewicz under this line [AB&JJJ].